On the self-force in Bopp-Podolsky electrodynamics

Jonathan Gratus[1]‡ , **Volker Perlick**[2]§ and **Robin W. Tucker**[1]‖

[1]Physics Department, Lancaster University, Lancaster LA1 4YB, UK; and The Cockcroft Institute, Warrington WA4 4AD, UK

[2]ZARM, University of Bremen, 28359 Bremen, Germany

Abstract. In the classical vacuum Maxwell-Lorentz theory the self-force of a charged point particle is infinite. This makes classical mass renormalization necessary and, in the special relativistic domain, leads to the Abraham-Lorentz-Dirac equation of motion possessing unphysical run-away and pre-acceleration solutions. In this paper we investigate whether the higher-order modification of classical vacuum electrodynamics suggested by Bopp, Landé, Thomas and Podolsky in the 1940s, can provide a solution to this problem. Since the theory is linear, Green-function techniques enable one to write the field of a charged point particle on Minkowski spacetime as an integral over the particle's history. By introducing the notion of timelike worldlines that are "bounded away from the backward light-cone" we are able to prescribe criteria for the convergence of such integrals. We also exhibit a timelike worldline yielding singular fields on a lightlike hyperplane in spacetime. In this case the field is mildly singular at the event where the particle crosses the hyperplane. Even in the case when the BoppPodolsky field is bounded, it exhibits a directional discontinuity as one approaches the point particle. We describe a procedure for assigning a value to the field on the particle worldline which enables one to define a finite Lorentz self-force. This is explicitly derived leading to an integro-differential equation for the motion of the particle in an external electromagnetic field. We conclude that any worldline solutions to this equation belonging to the categories discussed in the paper have continuous 4-velocities.

Keywords: Self-force, Radiation reaction, Higher-order electrodynamics, Bopp-Podolsky theory, Stress-energy-momentum tensors, Lorentz force, Abraham-Lorentz-Dirac.

1. Introduction

For many applications it is reasonable to model moving charges in terms of classical charged point particles. In accelerator physics, for example, it is usually neither desirable nor feasible to model particle beams in terms of extended classical charged bodies or of quantum matter. Therefore a mathematically consistent theory of classical charged point particles is of high relevance.

‡ Email: j.gratus@lancaster.ac.uk
§ Email: perlick@zarm.uni-bremen.de
‖ Email: r.tucker@lancaster.ac.uk

Unfortunately, such a theory does not exist so far. Of course, there is no problem as long as we restrict to a classical charged *test* particle and neglect the particle's self-interaction. Then the equation of motion is just the relativistic generalization of Newton's equation of motion with the rate of change of particle momentum equated to the Minkowski (relativistic) Lorentz force in a given external field and everything is fine. If, however, the self-field is taken into account, the theory becomes pathological. According to the Maxwell-Lorentz theory in vacuo, the electromagnetic field of a point charge becomes infinite at the position of the charge, so the particle experiences an infinite self-force. This infinity is so bad that the field energy in an arbitrarily small ball around the source is infinite which leads to an infinite term in the equation of motion of a point charge. Dirac (1938) suggested to counter-balance this infinity by postulating that the point charge carries a negative infinite "bare mass" which leads to the Abraham-Lorentz-Dirac equation. Even if one accepts this ad-hoc idea of an infinite bare mass, the problem has not been solved. The Abraham-Lorentz-Dirac equation is known to possess unphysical behavior such as run-away solutions and pre-acceleration. Reviews of this dilemma, including detailed accounts of the history, can be found in the comprehensive monographs by Rohrlich (2007) and Spohn (2007).

Since the Maxwell-Lorentz theory with point charges in vacuo does not lead to a consistent equation of motion of charged point particles, one might think about modifying this theory. In the course of history at least two such modifications have been suggested, both motivated by the desire of solving the problem of an infinite self-force, namely the Born-Infeld theory and the Bopp-Podolsky theory. The Born-Infeld theory is by far the better known of the two. This theory, which was suggested by Born & Infeld (1934), modifies the source free Maxwell vacuum theory by introducing non-linearities containing a new hypothetical constant of Nature b with the dimension of a (magnetic) field strength. For $b \to \infty$ the Maxwell equations in vacuo are recovered. The fact that the Maxwell theory is very well verified by many experiments is in agreement with the Born-Infeld theory as long as b is sufficiently large. By contrast, the Bopp-Podolsky theory retains linearity but introduces higher-derivative terms proportional to a factor ℓ^2 where ℓ is a new hypothetical constant of Nature with the dimension of a length. Again, for $\ell \to 0$ the Maxwell-Lorentz theory is recovered. The Bopp-Podolsky theory was first suggested by Bopp (1940). It was independently rediscovered by Podolsky (1942). Both Bopp and Podolsky formulated their theory in terms of an action functional and then derived the field equation which is of fourth order in the electromagnetic potential. As noted by both Bopp and Podolsky, this fourth-order equation is equivalent to a pair of second-order equations in a certain gauge. If rewritten in this form, the Bopp-Podolsky field system coincides with those of a theory suggested by Landé & Thomas (1941). Similar to the Born-Infeld theory, the Bopp-Podolsky theory was first formulated as a classical field theory but with the intention of deriving a quantum version later. In particular, Podolsky pursued both the classical and the quantum aspects of the theory in several follow-up articles with different co-authors, see Podolsky & Kikuchi (1944), Podolsky & Kikuchi (1945) and Podolsky & Schwed (1948). In the present article we

are interested only in the classical theory.

In both the Born-Infeld theory and the Bopp-Podolsky theory the self-field is bounded for a *static* point charge, i.e., for a point charge that is at rest in some inertial system in Minkowski spacetime. This was shown already in the earliest articles on these theories. Moreover, in both theories for such a charge the field energy in a ball of radius R around the charge is finite, even in the limit $R \to \infty$. To the best of our knowledge in the Born-Infeld theory little is known regarding such finiteness for accelerated point charges. In the Bopp-Podolsky theory the only result about accelerated point charges that we are aware of is due to Zayats (2014), who showed that the self-force is finite for a uniformly accelerated particle on Minkowski spacetime.

It is the purpose of this paper to add some results on the finiteness of the self-force in the Bopp-Podolsky theory. In our view, these results give strong support to the idea that the Bopp-Podolsky theory provides a consistent theory of classical charged point particles including the self-force. In Section 2 we briefly review the basic field equations of the Bopp-Podolsky theory on Minkowski spacetime, emphasizing the fact that because of their linearity Green-function techniques can be used. In Section 3 we restrict the equations of Section 2 to the case where the source of the electromagnetic field is a point charge with a prescribed worldline on Minkowski spacetime. We discuss various ways of writing the field strength at a point *off the worldline* as an integral over the particle's history. As a first example, we treat the simple case of a point charge that is at rest in an inertial system. In Section 4 the general problem of assigning a value of the field strength *on the worldline* is discussed. This is a precursor to the formulation of an integro-differential equation for the motion of a point charge that accommodates its finite self-force in an external electromagnetic field. As a second example, we treat the case of a uniformly accelerated charge. In Section 5 we present our main results on the finiteness of the field and of the self-force for general motion. We show that the self-force is finite unless the worldline approaches the light-cone in the past in a very contrived manner. As a third example, we discuss such a pathological worldline where the self-field actually diverges on a lightlike hyperplane and the self-force diverges at one point on the worldline. However, we demonstrate that even in this case the singularity of the field is so mild that it does not cause a problem for the equation of motion. In Section 6 we discuss how the Abraham-Lorentz-Dirac equation comes about in a particular limit as $\ell \to 0$, after classical mass renormalization.

In the body of the paper we formulate the Bopp-Podolsky theory on Minkowski spacetime in an inertial coordinate system. However, we have added an appendix where we consider the Bopp-Podolsky theory on a *curved* spacetime in arbitrary coordinates. This allows us to derive the dynamical (Hilbert) electromagnetic stress-energy-momentum tensor of the theory using exterior calculus. The appendix also includes a derivation of the relativistic Lorentz force in the Bopp-Podolsky theory from this tensor which is crucial for our reasoning in the body of the paper.

2. Bopp-Podolsky theory

We consider a time and space oriented Minkowski spacetime with standard inertial coordinates $\boldsymbol{x} = (x^0, x^1, x^2, x^3)$, with metric tensor

$$g = \eta_{ab} dx^a \otimes dx^b \tag{1}$$

where $(\eta_{ab}) = \mathrm{diag}(-1, 1, 1, 1)$. In this article all tensor field components on Minkowski spacetime are with respect to the class of global parallel bases adapted to these coordinates. Members of this class are related by elements of the proper Lorentz group, $SO(3, 1)$. Here and in the following, Einstein's summation convention is used for latin indices which take values $0, 1, 2, 3$ and for greek indices which take values $1, 2, 3$. Latin indices are lowered and raised with η_{ab} and with its inverse η^{ab}, respectively. We use units in which the the speed of light c equal to 1.

The higher-order electrodynamics suggested by Bopp (1940) and, independently, by Podolsky (1942) is based on the gauge invariant action functional

$$S[A] = \int_{\mathcal{M}} \left(\frac{1}{16\pi} F^{ab} F_{ab} + \frac{\ell^2}{16\pi} \partial^c F^{ab} \partial_c F_{ab} - A_a j^a \right) d^4x \tag{2}$$

where $\mathcal{M}$ is some compact region of Minkowski spacetime yielding a finite $S[A]$. Here A_a is the electromagnetic potential

$$F_{ab} = \partial_a A_b - \partial_b A_a \tag{3}$$

is the electromagnetic field strength, j^a is a conserved current density 4-vector field, $\{\partial_a = \partial/\partial x^a\}$ and ℓ is a hypothetical new constant of nature with the dimension of a length. It proves expedient to derive the Bopp-Podolsky field equations in terms of smooth fields and a smooth current source on $\mathcal{M}$ and then discuss particular singular solutions associated with a source having support on a timelike worldline. This eliminates the need to perform variations of (2) in a distributional context. Note that, for deriving the field equations by variational methods, the action functional (2) can be equivalently replaced with

$$\tilde{S}[A] = \int_{\mathcal{M}} \left(\frac{1}{16\pi} F^{ab} F_{ab} + \frac{\ell^2}{8\pi} \partial_a F^{ab} \partial^c F_{cb} - A_a j^a \right) d^4x \tag{4}$$

because the integrands differ only by a total divergence.

The field equations of the Bopp-Podolsky theory result from varying the action functional (2) or (4) with respect to the potential. They read

$$\partial^b F_{ba} - \ell^2 \Box \partial^b F_{ba} = -4\pi j_a \tag{5}$$

or, in terms of the potential in the Lorenz gauge $\partial^b A_b = 0$,

$$\Box A_a - \ell^2 \Box^2 A_a = -4\pi j_a \tag{6}$$

where $\Box = \partial^b \partial_b$ is the wave operator. Field equations involving the operator $\Box^2$ have also been investigated by Pais & Uhlenbeck (1950), cf. Pavlopoulos (1967).

Both Bopp and Podolsky observed that the fourth-order differential equation (6) can be reduced to a pair of second-order differential equations. More precisely, (6) is equivalent to

$$\Box \hat{A}_a = -4\pi j_a \,, \qquad \partial_a \hat{A}^a = 0 \,, \tag{7}$$

$$\Box \tilde{A}_a - \ell^{-2} \tilde{A}_a = -4\pi j_a \,, \qquad \partial_a \tilde{A}^a = 0 \,. \tag{8}$$

This can be demonstrated in the following way. Assume we have a solution A_a to (6). Then we define

$$\hat{A}_a = A_a - \ell^2 \Box A_a \,, \qquad \tilde{A}_a = -\ell^2 \Box A_a \tag{9}$$

and it is readily verified that (7) and (8) are indeed true. Conversely, assume that we have solutions $\hat{A}_a$ and $\tilde{A}_a$ to (7) and (8), respectively. Then we define

$$A_a = \hat{A}_a - \tilde{A}_a \tag{10}$$

and it is readily verified that (6) is true. This gives a one-to-one relation between solutions to (6) and pairs of solutions to (7) and (8) which allows one to view the Bopp-Podolsky theory as equivalent to a theory based on the two equations (7) and (8). The latter was suggested, shortly after Bopp but independently of him and shortly before Podolsky, by Landé & Thomas (1941). In a quantized version of the Landé-Thomas theory, (7) describes the usual (massless) photon while (8) describes a hypothetical "massive photon" whose Compton wave length is equal to the new constant of nature ℓ.

To find the retarded solution to the fourth-order Bopp-Podolsky equations (6), for any given divergence-free source j_a, we use the reduction to the second-order equations (7) and (8) and then apply standard Green-function techniques. The details were worked out by Landé and Thomas, see Section 9 in (Landé & Thomas 1941). The retarded solution to

$$\Box \hat{G}(\boldsymbol{x}-\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{11}$$

is

$$\hat{G}(\boldsymbol{x}-\boldsymbol{y}) = \begin{cases} (2\pi)^{-1}\delta(D(\boldsymbol{x}-\boldsymbol{y})^2) & \text{if } \boldsymbol{y} < \boldsymbol{x} \,, \\ 0 & \text{otherwise} \,, \end{cases} \tag{12}$$

and the retarded solution to

$$\Box \tilde{G}(\boldsymbol{x}-\boldsymbol{y}) - \ell^{-2} \tilde{G}(\boldsymbol{x}-\boldsymbol{y}) = -\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{13}$$

is

$$\tilde{G}(\boldsymbol{x}-\boldsymbol{y}) = \begin{cases} (2\pi)^{-1}\delta(D(\boldsymbol{x}-\boldsymbol{y})^2) - \dfrac{J_1(\ell^{-1}D(\boldsymbol{x}-\boldsymbol{y}))}{4\pi\ell D(\boldsymbol{x}-\boldsymbol{y})} & \text{if } \boldsymbol{y} < \boldsymbol{x} \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{14}$$

Here and in the following, J_n is the nth order Bessel function of the first kind, $\boldsymbol{y} < \boldsymbol{x}$ means that $\boldsymbol{y}$ is in the chronological past of $\boldsymbol{x}$ and $D(\boldsymbol{x}-\boldsymbol{y})$ is the Lorentzian distance between these two events,

$$D(\boldsymbol{x}-\boldsymbol{y}) \,=\, \sqrt{-\eta_{ab}(x^a-y^a)(x^b-y^b)}\,. \tag{15}$$

Hence, the retarded solution to (7) is

$$\hat{A}_a(x) \,=\, 4\pi \int_{\mathbb{R}^4} \hat{G}(\boldsymbol{x}-\boldsymbol{y})\, j_a(\boldsymbol{y})\, d^4\boldsymbol{y}\,, \tag{16}$$

the retarded solution to (8) is

$$\tilde{A}_a(\boldsymbol{x}) \,=\, 4\pi \int_{\mathbb{R}^4} \tilde{G}(\boldsymbol{x}-\boldsymbol{y})\, j_a(\boldsymbol{y})\, d^4\boldsymbol{y}\,, \tag{17}$$

and the retarded solution to (6) is

$$\begin{aligned} A_a(\boldsymbol{x}) &= 4\pi\int_{\mathbb{R}^4} \Big(\hat{G}(\boldsymbol{x}-\boldsymbol{y}) - \tilde{G}(\boldsymbol{x}-\boldsymbol{y})\Big)\, j_a(\boldsymbol{y})\, d^4\boldsymbol{y} \\ &= \int_{\boldsymbol{y}<\boldsymbol{x}} \frac{J_1(\ell^{-1}D(\boldsymbol{x}-\boldsymbol{y}))}{\ell D(\boldsymbol{x}-\boldsymbol{y})}\, j_a(\boldsymbol{y})\, d^4\boldsymbol{y}\,. \end{aligned} \tag{18}$$

The Lorenz gauge condition is satisfied by $\hat{A}_a$, $\tilde{A}_a$ and A_a when the current density satisfies the continuity equation

$$\partial_a j^a \,=\, 0\,. \tag{19}$$

Mathematically $\hat{A}_a$ is the retarded potential of the standard Maxwell theory. For $\ell \to 0$ we have $\tilde{G} \to 0$ and hence $\tilde{A}_a \to 0$, so in this limit the standard Maxwell theory is recovered, as is obvious from (2).

Equation (18) can be viewed as a map that assigns to each current density $\boldsymbol{j}$ the corresponding retarded Bopp-Podolsky potential $\boldsymbol{A}$. A general framework for investigating the question of whether this map is well-defined would be to assume that $\boldsymbol{j}$ is a (tempered) distribution and to ask if $\boldsymbol{A}$ is again a (tempered) distribution. As the Green function $\hat{G}-\tilde{G}$ does not satisfy a fall-off condition in all spacetime directions, this is a non-trivial question. In this paper we restrict to a more specific question. We assume that $\boldsymbol{j}$ is the current density associated with a point charge, and investigate whether the integral on the right-hand side of (18) converges for events $\boldsymbol{x}$ in the future of the worldline of the charge. If this is the case, the left-hand side of (18) is, of course, well defined, not only as a distribution but even as a function.

Up to now we have discussed only how the electromagnetic field can be calculated from its source, i.e., from the current that generates the field. We also need the equation for the Minkowski Lorentz force density which arises from the divergence of an electromagnetic stress-energy-momentum tensor T_{ab}. In inertial coordinates on Minkowski spacetime, it reads

$$\partial^c T_{ca} = F_{ab}\, j^b\,. \tag{20}$$

This equation defines the Minkowski force density that the field F_{ab} exerts on the current j^b. In the case that the current is concentrated on a worldline, it gives the self-force

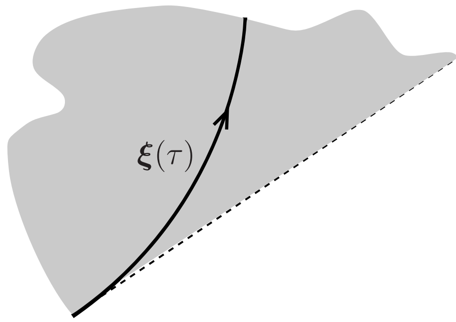

Figure 1. When the future pointing timelike worldline $\boldsymbol{\xi}(\tau)$ approaches a lightlike line (shown dashed) as $\tau \to \tau_{\min}$, its future is the subset of Minkowski spacetime, partially shown in grey above, bounded by a lightlike hyperplane containing the dashed line. The grey domain only consists of events which can be reached from the worldline along a future-oriented timelike curve.

in terms of field components on the worldline. A derivation of (20) in terms of the stress-energy-momentum tensor T_{ab} associated with the Bopp-Podolsky theory, is given in the Appendix.

3. The field of a point charge off the worldline

Let $\boldsymbol{\xi}(\tau) = (\xi^0(\tau), \xi^1(\tau), \xi^2(\tau), \xi^3(\tau))$ be an inextendible timelike C^∞ curve parametrized by proper time,

$$\dot{\xi}^a(\tau)\,\dot{\xi}_a(\tau) \;=\; -1\,. \tag{21}$$

Inextendible curves have parametrizations with τ belonging to an open interval $]\,\tau_{\min}, \tau_{\max}\,[$ where $\tau_{\min} \in \mathbb{R} \cup \{-\infty\}$ and $\tau_{\max} \in \mathbb{R} \cup \{+\infty\}$.¶

Consider the future of $\boldsymbol{\xi}(\tau)$, i.e., the set of all events that can be reached from the worldline along a future-oriented timelike curve. If the worldline approaches a light-cone asymptotically for $\tau \to \tau_{\min}$, its future is bounded by a lightlike hyperplane, see Fig. 1; otherwise it is all of $\mathbb{R}^4$. To each $\boldsymbol{x}$, in the future of $\boldsymbol{\xi}(\tau)$, but not on the worldline, we assign the retarded time $\tau_R(\boldsymbol{x})$, defined by the properties that

$$\Big(x^a - \xi^a(\tau_R(\boldsymbol{x}))\Big)\Big(x_a - \xi_a(\tau_R(\boldsymbol{x}))\Big) = 0\,, \qquad x^0 > \xi^0(\tau_R(\boldsymbol{x}))\,, \tag{22}$$

and introduce the retarded distance

$$r_R(\boldsymbol{x}) \;=\; -\,\dot{\xi}^a(\tau_R(\boldsymbol{x}))\Big(x_a - \xi_a(\tau_R(\boldsymbol{x}))\Big)\,. \tag{23}$$

Unless otherwise specified, in the following, all events $\boldsymbol{x}$ will belong to the future of the worldline. However, whether such events lie on or off the worldline will be made explicit. For events $\boldsymbol{x}$ off the worldline one has

$$x^a \;=\; \xi^a(\tau_R(\boldsymbol{x})) \;+\; r_R(\boldsymbol{x})\Big(\dot{\xi}^a(\tau_R(\boldsymbol{x})) + n^a(\boldsymbol{x})\Big) \tag{24}$$

¶ Some curves with unbounded acceleration reach past or future infinity in a finite proper time.

with a well-defined spatial unit vector $\boldsymbol{n}(\boldsymbol{x})$,

$$n^a(\boldsymbol{x})n_a(\boldsymbol{x}) \,=\, 1\,, \qquad \dot{\xi}^a(\tau_R(\boldsymbol{x}))n_a(\boldsymbol{x}) \,=\, 0\,, \tag{25}$$

see Fig. 2. By differentiation, we find

$$\partial_b\tau_R(\boldsymbol{x}) \,=\, -\dot{\xi}_b(\tau_R(\boldsymbol{x})) \,-\, n_b(\boldsymbol{x}) \tag{26}$$

and

$$\partial_b r_R(\boldsymbol{x}) \,=\, n_b(\boldsymbol{x}) \,+\, r_R(\boldsymbol{x})\ddot{\xi}^a(\tau_R(\boldsymbol{x}))n_a(\boldsymbol{x})\Big(\dot{\xi}_b(\tau_R(\boldsymbol{x})) + n_b(\boldsymbol{x})\Big)\,. \tag{27}$$

For a sequence of events $\boldsymbol{x}_N$ that approaches the worldline, $\boldsymbol{x}_N \to \boldsymbol{\xi}(\tau_0)$ as $N \to \infty$, $\tau_R(\boldsymbol{x}_N) \to \tau_0$ and $r_R(\boldsymbol{x}_N) \to 0$. The limits for $\boldsymbol{n}(\boldsymbol{x}_N)$, $\partial_b\tau_R(\boldsymbol{x}_N)$ and $\partial_b r_R(\boldsymbol{x}_N)$ do not exist as $N \to \infty$.

We model a point charge with worldline $\boldsymbol{\xi}(\tau)$ by the distributional current density

$$j_a(\boldsymbol{x}) \,=\, q\int_{\tau_{\min}}^{\tau_{\max}} \delta(\boldsymbol{x} - \boldsymbol{\xi}(\tau))\,\dot{\xi}_a(\tau)\,d\tau \tag{28}$$

where q is its electric charge. Then (16) gives the standard Liénard-Wiechert potential,

$$\hat{A}_a(\boldsymbol{x}) \,=\, \frac{q\,\dot{\xi}_a(\tau_R(\boldsymbol{x}))}{r_R(\boldsymbol{x})}\,, \tag{29}$$

and (18) reads

$$A_a(\boldsymbol{x}) \,=\, \frac{q}{\ell}\int_{\tau_{\min}}^{\tau_R(\boldsymbol{x})} \frac{J_1\Big(\ell^{-1}D(\boldsymbol{x} - \boldsymbol{\xi}(\tau))\Big)}{D(\boldsymbol{x} - \boldsymbol{\xi}(\tau))}\,\dot{\xi}_a(\tau)\,d\tau\,. \tag{30}$$

Both (29) and (30) are defined for events $\boldsymbol{x}$ off the worldline. In Section 4 we investigate what happens to the potentials and fields as $\boldsymbol{x}$ approaches the worldline.

Note that, in contrast to the Liénard-Wiechert potential (29), the potential (30) at an event $\boldsymbol{x}$ depends on the whole history of the charge from $\tau = \tau_{\min}$ up to $\tau = \tau_R(\boldsymbol{x})$. (The same is true, in general, in the standard Maxwell-Lorentz theory with point charges on curved spacetimes.) The integrand in (30) is bounded for $\tau \to \tau_R(\boldsymbol{x})$ because

$$\frac{J_1\Big(\ell^{-1}D(\boldsymbol{x} - \boldsymbol{\xi}(\tau))\Big)}{D(\boldsymbol{x} - \boldsymbol{\xi}(\tau))}\,\dot{\xi}_a(\tau) \;\longrightarrow\; \frac{\dot{\xi}_a(\tau_R(\boldsymbol{x}))}{2\ell} \qquad \text{for} \quad \tau \to \tau_R(\boldsymbol{x}) \tag{31}$$

where we have used the Bernoulli-l'Hôpital rule and $J_1'(0) = 1/2$. By contrast, for $\tau \to \tau_{\min}$, the individual components $\dot{\xi}_a(\tau)$ may blow up arbitrarily. Therefore, the existence of the integral on the right-hand side of (30) is not guaranteed. We show later that, for a fairly large class of worldlines, this integral does converge even absolutely, as a Lebesgue integral or as an improper Riemann integral, for all $\boldsymbol{x}$ in the future of the worldline; however, we also give a (contrived) example where it does *not* converge for some $\boldsymbol{x}$ in the future of the worldline.

On the assumption that the integral converges, at all events $\boldsymbol{x}$ we can differentiate (30) off the worldline with respect to x^b. Antisymmetrizing the resulting expression

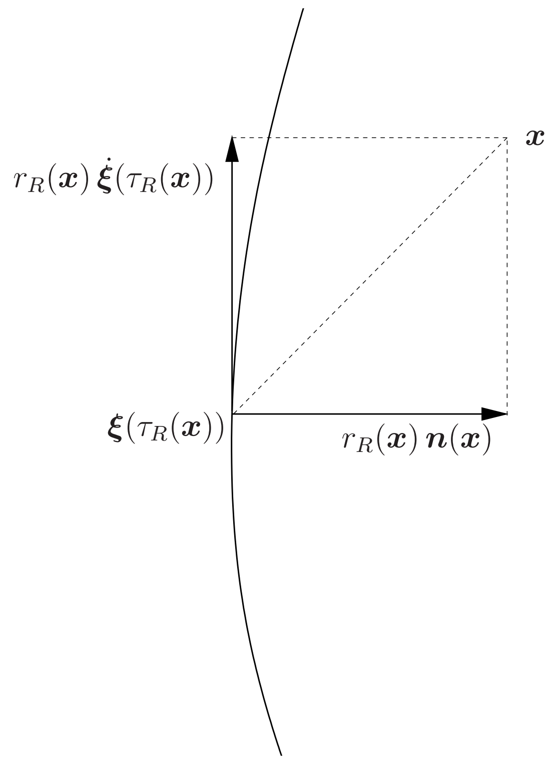

Figure 2. Retarded time and retarded distance

gives the field strength (3),

$$F_{ab}(\boldsymbol{x}) = \frac{q}{2\ell^2}\Big(\dot{\xi}_b(\tau_R(\boldsymbol{x}))n_a(\boldsymbol{x}) - \dot{\xi}_a(\tau_R(\boldsymbol{x}))n_b(\boldsymbol{x})\Big) \\ - \frac{q}{\ell^2}\int_{\tau_{\min}}^{\tau_R(\boldsymbol{x})} \frac{J_2\Big(\ell^{-1}D(\boldsymbol{x}-\boldsymbol{\xi}(\tau))\Big)}{D(\boldsymbol{x}-\boldsymbol{\xi}(\tau))^2}\Big((x_b-\xi_b(\tau))\dot{\xi}_a(\tau) - (x_a-\xi_a(\tau))\dot{\xi}_b(\tau)\Big)\,d\tau\;. \tag{32}$$

Here we have used (26) and the identities $2J_1(z) = z(J_0(z)+J_2(z))$ and $2J_1'(z) = J_0(z)-J_2(z)$ of the Bessel functions. Again, we postpone the discussion of what happens if $\boldsymbol{x}$ approaches the worldline to Section 4.

We observe that, keeping $\boldsymbol{x}$ fixed, we may use $\zeta = D(\boldsymbol{x}-\boldsymbol{\xi}(\tau))$ as the parameter along the worldline. Indeed, differentiation of the equation

$$\zeta^2 \;=\; -\,(x^a-\xi^a(\tau))(x_a-\xi_a(\tau)) \tag{33}$$

yields

$$\zeta\,d\zeta \;=\; \dot{\xi}^a(\tau)(x_a-\xi_a(\tau))d\tau. \tag{34}$$

As, by the reverse Schwartz inequality for timelike future-oriented vectors,

$$\dot{\xi}^a(\tau)(x_a-\xi_a(\tau)) < -\zeta < 0 \qquad \text{for} \quad 0<\zeta<\infty\;, \tag{35}$$

ζ is monotonically decreasing along the worldline. This guarantees that the equation $\zeta = D(\boldsymbol{x}-\boldsymbol{\xi}(\tau))$ can be solved for τ,

$$\zeta = D(\boldsymbol{x}-\boldsymbol{\xi}(\tau)) \qquad \Longleftrightarrow \qquad \tau = \varpi(\zeta,\boldsymbol{x})\;. \tag{36}$$

As proper time τ runs from $\tau_{\min}$ to $\tau_R(x)$, the new parameter ζ runs (backwards) from ∞ to 0. Hence at all events $\boldsymbol{x}$ (30) can be rewritten as

$$A_a(\boldsymbol{x}) \;=\; -\,\frac{q}{\ell}\int_0^\infty \left.\frac{J_1(\zeta/\ell)\,\dot{\xi}_a(\tau)}{\dot{\xi}^b(\tau)(x_b-\xi_b(\tau))}\right|_{\tau=\varpi(\zeta,\boldsymbol{x})} d\zeta\;. \tag{37}$$

Note that, if we regard $\boldsymbol{x}$ as a parameter, (37) has the form of a Hankel transform which transforms a function of ζ to a function of $1/\ell$.

We may use the parameter ζ in the formula for the field strength as well. If such a change of the integration variable is performed on the right-hand side of (32), the resulting equation reads

$$\begin{aligned} F_{ab}(\boldsymbol{x}) = {} & \frac{q}{2\ell^2}\Big(\dot{\xi}_b(\tau_R(\boldsymbol{x}))n_a(\boldsymbol{x})-\dot{\xi}_a(\tau_R(\boldsymbol{x}))n_b(\boldsymbol{x})\Big) \\ & +\,\frac{q}{\ell^2}\int_0^\infty \left.\frac{\Big((x_b-\xi_b(\tau))\dot{\xi}_a(\tau)-(x_a-\xi_a(\tau))\dot{\xi}_b(\tau)\Big)}{\dot{\xi}^c(\tau)(x_c-\xi_c(\tau))}\right|_{\tau=\varpi(\zeta,\boldsymbol{x})}\frac{J_2(\zeta/\ell)\,d\zeta}{\zeta}\;. \end{aligned} \tag{38}$$

An alternative expression for the field strength is obtained if, off the worldline, we differentiate (37) with respect to x^b and antisymmetrize,

$$\begin{aligned} F_{ab}(\boldsymbol{x}) = {} & \frac{q}{\ell}\int_0^\infty \left.\frac{J_1(\zeta/\ell)\Big(\ddot{\xi}_a(\tau)(x_b-\xi_b(\tau))-\ddot{\xi}_b(\tau)(x_a-\xi_a(\tau))\Big)}{\Big(\dot{\xi}^c(\tau)(x_c-\xi_c(\tau))\Big)^2}\right|_{\tau=\varpi(\zeta,\boldsymbol{x})} d\zeta \\ & -\,\frac{q}{\ell}\int_0^\infty \left.\frac{J_1(\zeta/\ell)\Big(\dot{\xi}_a(\tau)(x_b-\xi_b(\tau))-\dot{\xi}_b(\tau)(x_a-\xi_a(\tau))\Big)\Big(1+\ddot{\xi}^d(\tau)(x_b-\xi_b(\tau))\Big)}{\Big(\dot{\xi}^c(\tau)(x_c-\xi_c(\tau))\Big)^3}\right|_{\tau=\varpi(\zeta,\boldsymbol{x})} d\zeta\;. \end{aligned} \tag{39}$$

Alternatively (39) can be derived directly from (38) by integrating its second term by parts.

It is worth noting from (37) that we can derive another form of the potential by performing an integration by parts and using the identity $-J_1 = J_0'$ of Bessel functions. The resulting equation

$$\begin{aligned} A_a(\boldsymbol{x}) = {} & \frac{q\,\dot{\xi}_a(\tau_R(\boldsymbol{x}))}{r_R(\boldsymbol{x})} \\ & -\,q\int_0^\infty \left.\frac{\ddot{\xi}_a(\tau)\;-\;\dfrac{\dot{\xi}_a(\tau)\Big(1+\ddot{\xi}^d(\tau)(x_d-\xi_d(\tau))\Big)}{\dot{\xi}^c(\tau)(x_c-\xi_c(\tau))}}{\Big(\dot{\xi}^b(\tau)(x_b-\xi_b(\tau))\Big)^2}\right|_{\tau=\varpi(\zeta,\boldsymbol{x})} J_0(\zeta/\ell)\,\zeta\,d\zeta \end{aligned} \tag{40}$$

gives the deviation of the potential from the Liénard-Wiechert potential. With the help of standard asymptotic formulas for the Bessel function J_0 the right-hand side can be written as a power series in ℓ. Such asymptotic (i.e., in general non-convergent) expansions have been used, e.g., by Frenkel (1996) (also see Frenkel & Santos (1999)) and Zayats (2014).

Example 1: Charge at rest

The simplest case one can consider is a charge at rest in an appropriately chosen inertial system; this is equivalent to saying that the worldline of the charge is a straight timelike line,

$$\xi^a(\tau) = V^a\,\tau \tag{41}$$

with a constant four-vector $\boldsymbol{V}$ satisfying $V_a V^a = -1$. In this case (33) is a quadratic equation,

$$\zeta^2 \,=\, \tau^2 - 2\,\tau(r_R(\boldsymbol{x}) - \tau_R(\boldsymbol{x})) \,+\, 2\,\tau_R(\boldsymbol{x}) r_R(\boldsymbol{x}) + \tau_R(\boldsymbol{x})^2\,, \tag{42}$$

which can be easily solved for τ,

$$\varpi(\zeta,\boldsymbol{x}) \,=\, r_R(\boldsymbol{x}) \,+\, \tau_R(\boldsymbol{x}) \,-\, \sqrt{\zeta^2 + r_R(\boldsymbol{x})^2}\,. \tag{43}$$

Then (37) simplifies to

$$A_a(\boldsymbol{x}) \,=\, -\,\frac{q\,V^a}{\ell}\int_0^\infty \frac{J_1(\zeta/\ell)\,d\zeta}{\sqrt{\zeta^2 + r_R(\boldsymbol{x})^2}} \,=\, \frac{q\,V_a}{r_R(\boldsymbol{x})}\Big(1 - e^{-r_R(\boldsymbol{x})/\ell}\Big) \tag{44}$$

which is finite for all $\boldsymbol{x}$,

$$A_a(\boldsymbol{x}) \,\to\, -\,\frac{q\,V_a}{\ell} \qquad \text{for} \quad r_R(\boldsymbol{x}) \to 0\,. \tag{45}$$

By differentiation of (44) off the worldline, or equivalently by evaluation of (38) or (39) off the worldline, we get the field strength

$$\begin{aligned} F_{ab}(\boldsymbol{x}) \,&=\, \frac{q}{r_R(\boldsymbol{x})^2}\Big(n_a(\boldsymbol{x})V_b - n_b(\boldsymbol{x})V_a\Big)\left(1 \,-\, e^{-r_R(\boldsymbol{x})/\ell} - \frac{r_R(\boldsymbol{x})}{\ell}\,e^{-r_R(\boldsymbol{x})/\ell}\right) \\ &=\, \frac{q}{2\ell^2}\Big(n_a(\boldsymbol{x})V_b - n_b(\boldsymbol{x})V_a\Big)\Big(1 \,+\, O(r_R(\boldsymbol{x}))\Big)\,. \end{aligned} \tag{46}$$

This expression is not defined on the worldline; if a point on the worldline is approached, the limit of some components depend on the direction and we say that the field displays a *directional singularity*. Although for events $\boldsymbol{x}$ on the worldline the field (38) is undefined, the field (39) can be evaluated there and yields $F_{ab}(\boldsymbol{x}) = 0$ for all a, b. This value also arises by a certain averaging procedure described in the next section.

In the rest system of the charge we have $V_\mu = 0$ for $\mu = 1, 2, 3$ and $r_R(\boldsymbol{x}) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$ is just the ordinary radius coordinate. In this coordinate system (46) gives a radial electrostatic field with *modulus*

$$E(r) \,=\, \frac{q}{r^2}\left(1 \,-\, e^{-r/\ell} - \frac{r}{\ell}\,e^{-r/\ell}\right) \,=\, \frac{q}{2\ell^2}\Big(1 + O(r)\Big)\,. \tag{47}$$

In contrast to the Coulomb field of the standard Maxwell theory, the Bopp-Podolsky $E(r)$ stays finite as the worldline is approached. This result played a crucial role in the original work of Bopp (1940) and Podolsky (1942). It has the consequence that, at least for a charge at rest, the total field energy[+] is finite. Note, however, that the electric *vector field* cannot be continuously extended into the origin, because of the above-mentioned directional singularity, see Fig. 3.

[+] The total field energy $\mathcal{E} = \int_{\mathbf{R}^3} T_{00} d^3x$ where T_{ab} is defined in the appendix.

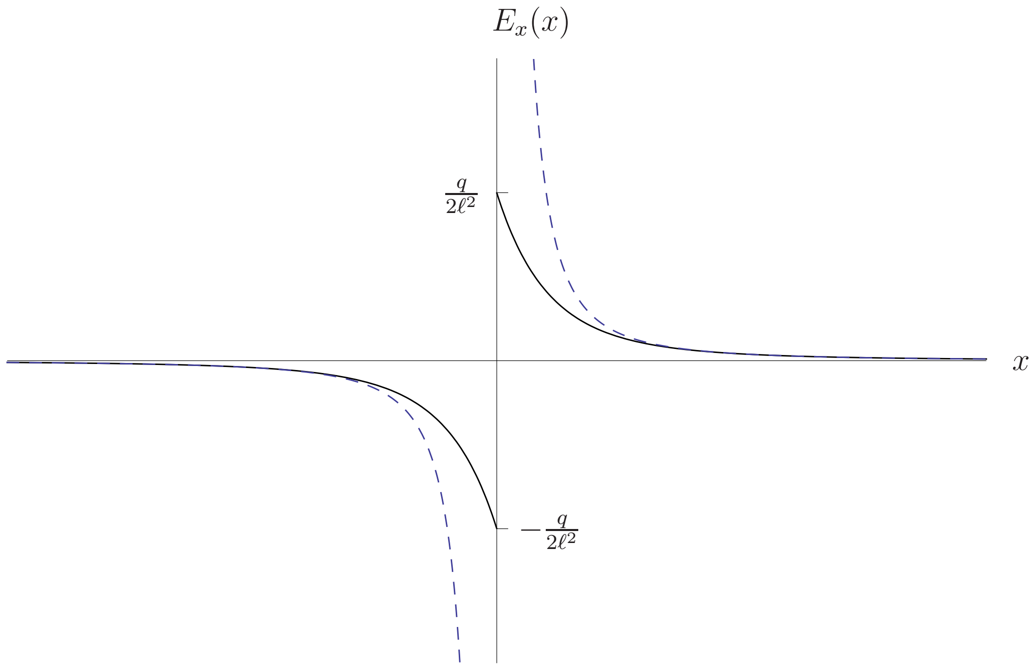

Figure 3. The x-component of the electric field strength on the x-axis of a charge at rest in the Bopp-Podolsky theory (solid) and in the standard Maxwell theory (dashed)

4. Field of a point charge on the worldline and self-force

The two expressions (30) and (37) for the potential are equivalent off the worldline, and so are the two expressions (38) and (39) for the field strength. We now discuss their behavior on and near the worldline. This is crucial because the value of the field strength on the worldline will determine the self-force. Both representations of the potential (30) and (37) are well defined and continuous on the worldline. If the potential were differentiable, its derivative would give the field strength on the worldline without any ambiguity. However, it is *not* differentiable on the worldline. This is the reason why the expression (38), which results from differentiating (30), and the expression (39), which results from differentiating (37), behave differently on the worldline.

First observe that (38) is not defined at events on the worldline because it involves the derivative $\partial_b \tau_R(\boldsymbol{x})$ which is not defined at such events. By contrast, (39) does not involve this derivative and gives a *unique* value for the field strength on the worldline, provided that the integrals converge. However, this value does not result from differentiating the potential on the worldline because in order to differentiate (37) one must pass the derivative under the integral; this is only valid if the integrand has a continuous derivative, which does not occur if $\boldsymbol{x}$ is on the worldline.

As noted, the field $F_{ab}(\boldsymbol{x})$ given by (38) or (39) is not continuous in a neighborhood of events on the worldline: if a point $\boldsymbol{x}$ on the worldline is approached some of the components have non-zero *finite* limits that depend on the direction taken. This is manifest in (38) where the discontinuity arises solely from the first term. Such behavior contrasts with similar problems in defining the self-force on a charged particle

in Maxwell-Lorentz electrodynamics where a similar directional dependence arises as one approaches the particle worldline. However, in that theory the same limiting values are infinite.

Since the self-force (to be defined below) depends on the values of $F_{ab}(\boldsymbol{x})$ on the worldline, one must assign them *specific* values. Such assignments should be based on physical criteria beyond the mathematical analysis thus far. For example an isolated free point charge should remain in "inertial" motion in the absence of external fields, i.e. the self-force should be zero. From example 1, this can be achieved by assigning the value zero to all $F_{ab}(\boldsymbol{x})$ on the inertial worldline in this case. This assignment is equivalent to either adopting (39) directly for $F_{ab}(\boldsymbol{x})$ or applying a "directional-averaging" (see below) to (38). Although for arbitrary motion one may always define a directional-averaging such that the value given by (39) arises from such a process applied to (38), this is by no means a unique procedure. In general one may consider more complex averaging procedures involving constructions based on extensions of the natural Frenet frame defined by the worldline.

A procedure based solely on the worldline's tangent vector, rather than the full geometry of its Frenet frame can be done most easily if we introduce the (spatial) *retarded 2-sphere*, see Fig. 4,

$$S(\tau_0, r_0) = \{\boldsymbol{x} \in \mathbb{R}^4 |\, \tau_R(\boldsymbol{x}) = \tau_0\,,\ r_R(\boldsymbol{x}) = r_0\}\,. \tag{48}$$

The directional-average of any spacetime tensor's inertial components $K^{a\cdots}{}_{b\ldots}(\boldsymbol{x})$ over the 2-sphere $S(\tau_0, r_0)$ is defined as

$$\overline{K^{a\cdots}{}_{b\ldots}} = \frac{1}{4\pi}\iint_{S(\tau_0,r_0)} K^{a\cdots}{}_{b\ldots}(\boldsymbol{x})\, dS \tag{49}$$

where dS is the natural surface measure on $S(\tau_0, r_0)$, induced by the ambient Minkowski metric. This integral depends only on τ_0, r_0, $\boldsymbol{\xi}(\tau_0)$ and $\dot{\boldsymbol{\xi}}(\tau_0)$ and is manifestly Lorentz *covariant* with respect to global Lorentz transformations. I.e. if

$$K^{a'\cdots}{}_{b'\ldots}(\boldsymbol{x}) = \Lambda^{a'}{}_{a}\cdots \Lambda_{b'}{}^{b}\cdots K^{a\cdots}{}_{b\ldots}(\boldsymbol{x}) \tag{50}$$

then

$$\overline{K^{a'\cdots}{}_{b'\ldots}} = \Lambda^{a'}{}_{a}\cdots \Lambda_{b'}{}^{b}\cdots \overline{K^{a\cdots}{}_{b\ldots}} \tag{51}$$

where $\Lambda^{a'}{}_{a} \in SO(3,1)$.

Coordinating $S(\tau_0, r_0)$ with spherical polars (ϑ, φ), the measure $dS = \sin\vartheta\, d\vartheta\, d\varphi$ and (49) becomes

$$\overline{K^{a\cdots}{}_{b\ldots}} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} K^{a\cdots}{}_{b\ldots}(\boldsymbol{x}) \sin\vartheta\, d\vartheta\, d\varphi \tag{52}$$

In terms of an orthonormal tetrad $(\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ with $\boldsymbol{e}_0 = \dot{\boldsymbol{\xi}}(\tau_0)$

$$n^a(\boldsymbol{x}) = \cos\varphi\sin\vartheta\, e_1^a + \sin\varphi\sin\vartheta\, e_2^a + \cos\vartheta\, e_3^a \tag{53}$$

for $\boldsymbol{x} \in S(\tau_0, r_0)$. Hence $\overline{n^a} = 0$. Since $\dot{\xi}^a(\tau_R(\boldsymbol{x})) = \dot{\xi}^a(\tau_0)$ is constant on $S(\tau_0, r_0)$ the first term on the right-hand side of (38) averages to zero. Hence its limit for $r_0 \to 0$ is

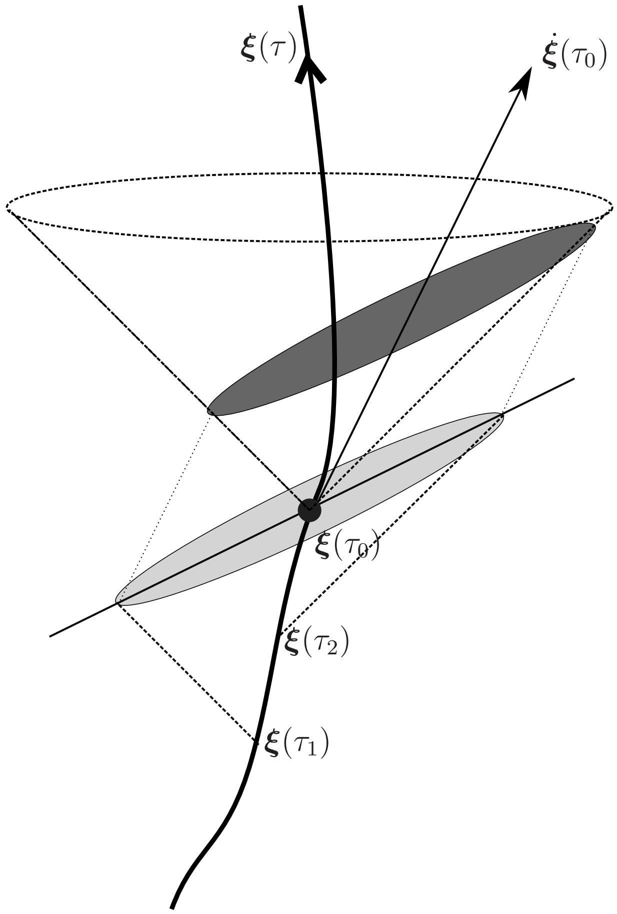

Figure 4. This figure illustrates a retarded 2-sphere (dark grey) and a rest 2-sphere (light grey), defined by an arbitrary timelike worldline in Minkowski spacetime. The rest sphere with origin $\boldsymbol{\xi}(\tau_0)$ lies in the orthocomplement of the tangent vector $\dot{\boldsymbol{\xi}}(\tau_0)$. The retarded sphere is the intersection of the rest sphere's parallel translation along $\dot{\boldsymbol{\xi}}(\tau_0)$ with the forward lightcone of $\boldsymbol{\xi}(\tau_0)$. The averaging procedure described in the text corresponds to sampling the fields of the particle as the radii of these spheres tend to zero. Fields on the rest sphere are generated by events on the past history of $\boldsymbol{\xi}(\tau_2)$. By contrast fields on the retarded sphere are all generated by the past history of $\boldsymbol{\xi}(\tau_0)$.

zero as well, so this term gives no contribution to the average of the field (38) at $\boldsymbol{\xi}(\tau_0)$. Since the second term in (38) is continuous in a neighborhood of the $\boldsymbol{\xi}(\tau_0)$ its average is well defined. Hence the averaged value of (38) equals the second term on the right hand side of (38) and is also equal to (39) after an integration by parts.

An alternative averaging procedure can be defined in terms of *rest* spheres in the orthocomplement of $\dot{\xi}(\tau_0)$, see Fig. 4. In this case $\dot{\xi}^a(\tau_R(\boldsymbol{x}))$ is *not* constant on a sphere of finite radius r_0, so the calculation is less convenient; however, in the limit of r_0 tending to zero one finds, again, that the first term on the right-hand side of (38) averages to zero. Thus these two averaging procedures give the same result and correspond to surrounding the worldline either by a *Bhabha tube* or by a *Dirac tube*. See Norton (2009) and Ferris & Gratus (2011) for a similar discussion in the context of the standard Maxwell-Lorentz theory with point charges.

Motivated by the application of this procedure to inertial motion in example 1 and the equivalence of the directional-average of (38) with (39) for general motion, we assign to $F_{ab}(\boldsymbol{x})$ on the worldline the unique value (39) for *all motions*. This is intuitively persuasive if one thinks of the point charge as the limiting case of an extended charge whose size tends to zero. Directional-averaging is formulated as an axiom in the living review on the self-force by Poisson, Pound and Vega, see Section 24.1 in (Poisson et al. 2011).

Given a definition of a *finite* field $F_{ab}(\boldsymbol{x})$ at an event $\boldsymbol{x} = \boldsymbol{\xi}(\tau_0)$ on the worldline, the self-force $f^{\mathrm{S}}_a(\tau_0)$ is defined as the relativistic Lorentz force* exerted by $F_{ab}(\boldsymbol{\xi}(\tau_0))$ on the point charge that produces the field:

$$f^{\mathrm{S}}_a(\tau_0) \;=\; q\, F_{ab}(\boldsymbol{\xi}(\tau_0))\, \dot{\xi}^b(\tau_0)\,. \tag{54}$$

The equation of motion for some C^0 functions $\xi^a(\tau)$ is assumed to take the form

$$m\, \ddot{\xi}_a(\tau) \;=\; f^{\mathrm{S}}_a(\tau) \,+\, f^{\mathrm{e}}_a(\tau) \tag{55}$$

where m denotes the *finite* inertial mass of the particle and $f^{\mathrm{e}}_a(\tau)$ is an external Minkowski force. If the latter is electromagnetic in origin, $f^{\mathrm{e}}_a(\tau) \,=\, q\, F^{\mathrm{e}}_{ab}(\boldsymbol{\xi}(\tau))\, \dot{\xi}^b(\tau)$, were F^{e}_{ab} solves the Bopp-Podolsky field equations with all sources other than the point particle with charge q, which includes the special case of no other sources. If $f^{\mathrm{e}}_a(\tau)$ is given, (55) is an integro-differential equation for the worldline $\boldsymbol{\xi}(\tau)$.

In the standard Maxwell-Lorentz theory with point charges, the self-force is infinite; therefore, it is necessary to perform a mass renormalization, introducing a "bare mass" of the particle that is negative infinite. By contrast, in the Bopp-Podolsky theory there is no need or justification for introducing an infinite bare mass.

From (39) the self-force reads

$$f^{\mathrm{S}}_a(\tau_0) \;=\; \frac{q^2 \dot{\xi}^b(\tau_0)}{\ell} \int_0^\infty \frac{\partial}{\partial\zeta} W_{ab}(\zeta, \boldsymbol{\xi}(\tau_0)) \frac{J_1(\zeta/\ell)\, d\zeta}{\zeta}\,. \tag{56}$$

where

$$W_{ab}(\zeta, \boldsymbol{\xi}(\tau_0)) \;=\; \left. \frac{(\xi_b(\tau_0) - \xi_b(\tau))\dot{\xi}_a(\tau) - (\xi_a(\tau_0) - \xi_a(\tau))\dot{\xi}_b(\tau)}{\dot{\xi}^c(\tau)(\xi_c(\tau_0) - \xi_c(\tau))} \right|_{\tau = \varpi(\zeta, \boldsymbol{\xi}(\tau_0))}\,. \tag{57}$$

Although (55) contains explicit derivatives of maximal order two, the presence of the integral (56) over the past history of the worldline implies that it cannot be solved given only $f^{\mathrm{e}}_a(\tau)$ and the initial position and velocity of the particle at any initial τ. Integro-differential equations involving retarded (or memory) effects are not uncommon in continuum mechanics and the Maxwell electrodynamics of continuous media. Indeed even in vacuo the modifications of the Abraham-Lorentz-Dirac equation due to the presence of a background gravitational field yields a similar integral over the past history of a charged point particle worldline (DeWitt & Brehme 1960). In such problems additional physical criteria motivate analytic or numerical procedures

* In the appendix, this force is shown to arise from the divergence of the electromagnetic stress-energy-momentum tensor associated with the Bopp-Podolsky theory.

that can be used to construct solutions. Since in principle the past history of any point charge is not empirically accessable, it seems inevitable that similar methodologies will be required to determine any future motion uniquely from (55) given only consistent field and particle data on some arbitrary 3-dimensional spacelike hypersurface in Minkowski spacetime. These general ideas will be developed elsewhere.

Example 2: Uniformly accelerated motion

For a particle in hyperbolic motion with constant acceleration a in its instantaneous rest frame, the worldline is given by

$$\begin{aligned} \xi^0(\tau) &= \frac{1}{a}\sinh(a\tau)\,, \qquad \xi^1(\tau) = \frac{1}{a}\cosh(a\tau) \\ \xi^2(\tau) &= \xi^3(\tau) = 0\,. \end{aligned} \tag{58}$$

In this case, for a point $\boldsymbol{x} = \boldsymbol{\xi}(\tau_0)$ on the worldline (33) reads

$$\zeta^2 = \frac{2}{a^2}\Big(\cosh(a(\tau_0 - \tau)) - 1\Big)\,. \tag{59}$$

The self-force (56) reduces to

$$f^{\mathrm{S}}_a(\tau_0) = -\,q^2\,\ddot{\xi}_a(\tau_0)\int_0^\infty \frac{J_2(\frac{\zeta}{\ell})\,d\zeta}{2\,\ell^2\sqrt{1+\frac{a^2\zeta^2}{4}}} \tag{60}$$

which can be expressed in terms of the Bessel functions I_1 and K_1,

$$f^{\mathrm{S}}_a(\tau_0) = -\,\frac{q^2}{a\ell^2}\,I_1((a\ell)^{-1})\,K_1((a\ell)^{-1})\,\ddot{\xi}_a(\tau_0)\,. \tag{61}$$

This result was recently found by Zayats (2014). So the self-force here is manifestly finite.

5. Finiteness of the field of a point charge and of the self-force

In the standard Maxwell-Lorentz theory, the (Liénard-Wiechert) potential of a point charge and the corresponding field strength are singular on the worldline of the source. By contrast, in the Bopp-Podolsky theory there is a large class of worldlines for which the self-force is given by an absolutely converging integral.

As $\dot{\boldsymbol{\xi}}(\tau)$ is a timelike unit vector, and $\boldsymbol{x} - \boldsymbol{\xi}(\tau)$ has Lorentz length ζ, we may write

$$\dot{\xi}^a(\varpi(\zeta,\boldsymbol{x})) = \cosh\chi(\zeta,\boldsymbol{x})\,\delta^a_0 + \sinh\chi(\zeta,\boldsymbol{x})\,\nu^\rho(\zeta,\boldsymbol{x})\,\delta^a_\rho \tag{62}$$

and

$$x^a - \xi^a(\varpi(\zeta,\boldsymbol{x})) = \zeta\,\Big(\cosh\psi(\zeta,\boldsymbol{x})\,\delta^a_0 + \sinh\psi(\zeta,\boldsymbol{x})\,\mu^\rho(\zeta,\boldsymbol{x})\,\delta^a_\rho\Big) \tag{63}$$

where $\boldsymbol{\nu}(\zeta,\boldsymbol{x})$ and $\boldsymbol{\mu}(\zeta,\boldsymbol{x})$ are spatial unit vectors,

$$\nu_\rho(\zeta,\boldsymbol{x})\,\nu^\rho(\zeta,\boldsymbol{x}) = \mu_\rho(\zeta,\boldsymbol{x})\,\mu^\rho(\zeta,\boldsymbol{x}) = 1\,. \tag{64}$$

We introduce the following terminology.

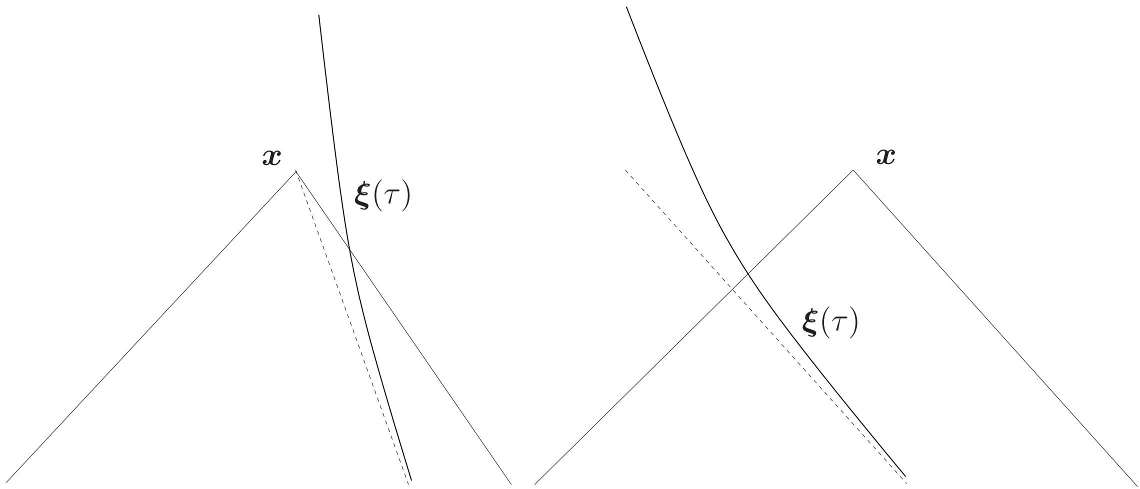

Figure 5. Worldline bounded away from the past light-cone of $\boldsymbol{x}$ (left) and not bounded away from the past light-cone of $\boldsymbol{x}$ (right)

Definition 1. The worldline $\boldsymbol{\xi}$ is *bounded away from the past light-cone* of an event $\boldsymbol{x}$ in the future of $\boldsymbol{\xi}$ if $\psi(\zeta, \boldsymbol{x})$ stays bounded for $\zeta \to \infty$.

Geometrically, $\boldsymbol{\xi}$ is not bounded away from the past light-cone of $\boldsymbol{x}$ if and only if there is a sequence τ_k such that $(\boldsymbol{x} - \boldsymbol{\xi}(\tau_k))/(x^0 - \xi^0(\tau_k))$ approaches a lightlike vector for $\tau_k \to \tau_{\min}$, see Fig. 5.

The notion of being bounded away from the light-cone implicitly refers to a particular inertial coordinate system chosen, and it refers to a particular event $\boldsymbol{x}$. However, the notion is actually independent of these choices, as the following proposition shows.

Proposition 1. The property of the worldline being bounded away from the past light-cone of $\boldsymbol{x}$ is preserved if we change the inertial coordinate system by an orthochronous Lorentz transformation. If this property is true for one event $\boldsymbol{x}$ in the future of the worldline, then the future of the worldline is all of $\mathbb{R}^4$ and the property is true for all other events $\boldsymbol{y} \in \mathbb{R}^4$ as well.

Proof. From (63) we read that

$$\frac{x^0 - \xi^0(\varpi(\zeta, \boldsymbol{x}))}{\sqrt{-\Big(x^a - \xi^a(\varpi(\zeta, \boldsymbol{x}))\Big)\Big(x_a - \xi_a(\varpi(\zeta, \boldsymbol{x}))\Big)}} = \cosh \psi(\zeta, \boldsymbol{x})\,. \tag{65}$$

The worldline is bounded away from the light-cone of $\boldsymbol{x}$ if and only if the right-hand side is bounded for $\zeta \to \infty$, i.e., if and only if there exists $\delta > 0$ such that

$$\frac{x^0 - \xi^0(\tau)}{\sqrt{-\Big(x^a - \xi^a(\tau)\Big)\Big(x_a - \xi_a(\tau)\Big)}} < \delta \tag{66}$$

for $\tau_{\rm min} < \tau < \tau_0$ with some τ_0. To prove the first part of the proposition, we assume that this condition holds in the chosen inertial system. Under a Lorentz transformation, $\tilde{x}^a = \Lambda^a{}_b x^b$, the denominator on the left-hand side of (66) is unchanged,

$$\Big(\tilde{x}^a - \tilde{\xi}^a(\tau)\Big)\Big(\tilde{x}_a - \tilde{\xi}_a(\tau)\Big) \;=\; \Big(x^a - \xi^a(\tau)\Big)\Big(x_a - \xi_a(\tau)\Big)\,, \tag{67}$$

while the numerator changes according to

$$\tilde{x}^0 - \tilde{\xi}^0(\tau) \;=\; \Lambda^0{}_0(x^0 - \xi^0(\tau)) + \Lambda^0{}_\mu(x^\mu - \xi^\mu(\tau))\,. \tag{68}$$

As $\boldsymbol{x} - \boldsymbol{\xi}(\tau)$ is timelike and future-pointing,

$$|x^\mu - \xi^\mu(\tau)| < x^0 - \xi^0(\tau) \tag{69}$$

for $\mu = 1, 2, 3$. As a consequence, (68) implies that

$$\tilde{x}^0 - \tilde{\xi}^0(\tau) < K\Big(x^0 - \xi^0(\tau)\Big) \tag{70}$$

with some positive constant K. Here we have assumed that the Lorentz transformation is orthochronous, $\Lambda^0{}_0 > 0$. From (66) we find that

$$\frac{\tilde{x}^0 - \tilde{\xi}^0(\tau)}{\sqrt{-\Big(\tilde{x}^a - \tilde{\xi}^a(\tau)\Big)\Big(\tilde{x}_a - \tilde{\xi}_a(\tau)\Big)}} \;<\; K\,\delta \tag{71}$$

for $\tau_{\rm min} < \tau < \tau_0$ which proves that the condition of the worldline being bounded away from the light-cone of the chosen event holds in the twiddled coordinate system as well. To prove the second part of the proposition, we observe that (63) implies

$$\frac{|\vec{x} - \vec{\xi}(\varpi(\zeta, \boldsymbol{x}))|}{x^0 - \xi^0(\varpi(\zeta, \boldsymbol{x}))} \;=\; |\tanh\psi(\zeta, \boldsymbol{x})|\,. \tag{72}$$

Here and in the following, we write $|\vec{a}| = \sqrt{\delta_{\mu\nu}a^\mu a^\nu}$ for any $\vec{a} = (a^1, a^2, a^3)$. The worldline is bounded away from the light-cone of $\boldsymbol{x}$ if and only if the right-hand side of (72) is bounded away from 1 for $\zeta \to \infty$, i.e., if and only if there is a λ with $0 < \lambda < 1$ such that

$$\frac{|\vec{x} - \vec{\xi}(\tau)|}{x^0 - \xi^0(\tau)} \;<\; \lambda \tag{73}$$

for $\tau_{\rm min} < \tau < \tau_0$ with some τ_0. Let us assume that this condition holds for some particular event $\boldsymbol{x}$. Let $\boldsymbol{y}$ be any other event and choose a constant μ such that $\lambda < \mu < 1$. Define

$$t := \frac{\mu y^0 - \lambda x^0 - |\vec{y} - \vec{x}|}{\mu - \lambda}\,. \tag{74}$$

Then we have, for all τ such that $\xi^0(\tau) < t$,

$$\begin{aligned} |\vec{y} - \vec{\xi}(\tau)| - \mu(y^0 - \xi^0(\tau)) &= |\vec{y} - \vec{x} + \vec{x} - \vec{\xi}(\tau)| - \mu(y^0 - \xi^0(\tau)) \\ &\le |\vec{y} - \vec{x}| + |\vec{x} - \vec{\xi}(\tau)| - \mu(y^0 - \xi^0(\tau)) \\ &< |\vec{y} - \vec{x}| + \lambda(x^0 - \xi^0(\tau)) - \mu(y^0 - \xi^0(\tau)) \\ &< |\vec{y} - \vec{x}| + \lambda x^0 - \mu y^0 + (\mu - \lambda)\, t \;=\; 0 \end{aligned} \tag{75}$$

hence

$$\frac{|\vec{y} - \vec{\xi}(\tau)|}{y^0 - \xi^0(\tau)} < \mu \tag{76}$$

for $\tau_{\min} < \tau < \hat{\tau}_0$ with some $\hat{\tau}_0$. This inequality demonstrates that $\boldsymbol{y}$ is in the future of the worldline and that the worldline is bounded away from the light-cone of $\boldsymbol{y}$ as well. □

Because of this result, we may simply say that a worldline is bounded away from the past light-cone, without any reference to a specific event $\boldsymbol{x}$.

We now show that the field of a point charge is finite if its worldline is bounded away from the past light-cone. Using the notation of (62) and (63), the potential (37) reads

$$A_a(\boldsymbol{x}) = -\frac{q}{\ell}\int_0^\infty \frac{\Big(\eta_{a0} \,+\, \tanh\chi(\zeta,\boldsymbol{x})\,\nu^\rho(\zeta,\boldsymbol{x})\,\eta_{a\rho}\Big)\; J_1(\zeta/\ell)\,d\zeta}{\cosh\psi(\zeta,\boldsymbol{x})\,\Big(1\,-\,\tanh\psi(\zeta,\boldsymbol{x})\tanh\chi(\zeta,\boldsymbol{x})\,\mu^\rho(\zeta,\boldsymbol{x})\,\nu_\rho(\zeta,\boldsymbol{x})\Big)\;\zeta}\,. \tag{77}$$

For expressing the field strength tensor at a chosen event $\boldsymbol{x}$ using the notation of (62) and (63), we may choose the inertial coordinate system such that $\dot{\xi}^a(\tau_R(\boldsymbol{x})) = \delta^a_0$. Then the electric and magnetic components of the field strength tensor (38) read, respectively,

$$F_{0\sigma}(\boldsymbol{x}) = \frac{q\,n_\sigma(\boldsymbol{x})}{2\ell^2} + \frac{q}{\ell^2}\int_0^\infty \frac{\Big(\tanh\psi(\zeta,\boldsymbol{x})\,\mu_\sigma(\zeta,\boldsymbol{x}) - \tanh\chi(\zeta,\boldsymbol{x})\,\nu_\sigma(\zeta,\boldsymbol{x})\Big) J_2(\zeta/\ell)\,d\zeta}{\Big(1\,-\,\tanh\psi(\zeta,\boldsymbol{x})\tanh\chi(\zeta,\boldsymbol{x})\,\mu^\rho(\zeta,\boldsymbol{x})\,\nu_\rho(\zeta,\boldsymbol{x})\Big)\;\zeta}\,, \tag{78}$$

$$F_{\rho\sigma}(\boldsymbol{x}) = \tag{79}$$
$$\frac{q}{\ell^2}\int_0^\infty \frac{\tanh\psi(\zeta,\boldsymbol{x})\tanh\chi(\zeta,\boldsymbol{x})\,\Big(\nu_\sigma(\zeta,\boldsymbol{x})\mu_\rho(\zeta,\boldsymbol{x}) - \mu_\sigma(\zeta,\boldsymbol{x})\nu_\rho(\zeta,\boldsymbol{x})\Big)\; J_2(\zeta/\ell)\,d\zeta}{\Big(1\,-\,\tanh\psi(\zeta,\boldsymbol{x})\tanh\chi(\zeta,\boldsymbol{x})\,\mu^\rho(\zeta,\boldsymbol{x})\,\nu_\rho(\zeta,\boldsymbol{x})\Big)\;\zeta}\,.$$

In a coordinate system with $\dot{\xi}^a(\tau_0) = \delta^a_0$ the self-force (56) is given by

$$f^{\mathrm{S}}_a(\tau_0) = \tag{80}$$
$$-\frac{q^2}{\ell^2}\,\eta_{a\sigma}\int_0^\infty \frac{\Big(\tanh\psi(\zeta,\boldsymbol{\xi}(\tau_0))\,\mu^\sigma(\zeta,\boldsymbol{\xi}(\tau_0)) - \tanh\chi(\zeta,\boldsymbol{\xi}(\tau_0))\,\nu^\sigma(\zeta,\boldsymbol{\xi}(\tau_0))\Big)\; J_2(\zeta/\ell)\,d\zeta}{\Big(1\,-\,\tanh\psi(\zeta,\boldsymbol{\xi}(\tau_0))\tanh\chi(\zeta,\boldsymbol{\xi}(\tau_0))\,\mu^\rho(\zeta,\boldsymbol{\xi}(\tau_0))\,\nu_\rho(\zeta,\boldsymbol{\xi}(\tau_0))\Big)\;\zeta}\,.$$

We can now prove the following result.

Proposition 2. If the worldline $\boldsymbol{\xi}$ is bounded away from the past light-cone, the integrals on the right-hand sides of (77), (78) and (79) are absolutely convergent for all $\boldsymbol{x} \in \mathbb{R}^4$; the integral on the right-hand side of (80) is absolutely convergent for all $\tau_{\min} < \tau_0 < \tau_{\max}$.

Proof. As the integrands in (77), (78), (79) and (80) stay finite for $\zeta \to 0$, we only have to verify that they fall off sufficiently quickly for $\zeta \to \infty$. We first observe that $\tanh\alpha$ increases from -1 to 1 if α varies from $-\infty$ to ∞, and that $|\mu^\rho\nu_\rho| \le 1$. Therefore,

the condition of ψ being bounded implies that $\tanh\psi \tanh\chi\, \mu^\rho \nu_\rho$ is bounded away from 1. Thus, in each of the four equations (77), (78), (79) and(80) the modulus of the integrand is bounded by a term of the form $K|J_k(\zeta/\ell)|/\zeta$ where K is independent of ζ and k is either 1 or 2. As $|J_k(\zeta/\ell)|$ falls off like $\zeta^{-1/2}$ for $\zeta \to \infty$, this guarantees absolute convergence of the integral. □

This proposition demonstrates that, in particular, the self-force is finite for a large class of worldlines. Actually, the requirement of the worldline being bounded away from the past light-cone is sufficient but not necessary for finiteness of the self-force. The following proposition shows that there is another class of worldlines, including ones which are *not* bounded away from the past light-cone, for which the self-force is finite.

Proposition 3. Assume that the worldline $\boldsymbol{\xi}$ is confined to a two-dimensional timelike plane P in Minkowski spacetime. Then the integral on the right-hand side of (80) is absolutely convergent for all $\tau_{\min} < \tau_0 < \tau_{\max}$.

Proof. As the worldline is in a timelike plane, the unit vectors $\nu_\sigma(\zeta, \boldsymbol{\xi}(\tau_0))$ and $\mu_\sigma(\zeta, \boldsymbol{\xi}(\tau_0))$ must be constant and equal or opposite, $\nu_\sigma = \pm\mu_\sigma$. Then (80) simplifies to

$$f^{\mathrm{S}}_a(\tau_0) \;=\; -\frac{q^2}{\ell^2}\,\eta_{a\sigma}\,\mu^\sigma \int_0^\infty \tanh\Big(\psi(\zeta, \boldsymbol{\xi}(\tau_0)) \mp \chi(\zeta, \boldsymbol{\xi}(\tau_0))\Big)\,\frac{J_2(\zeta/\ell)\,d\zeta}{\zeta}\,. \tag{81}$$

As $|\tanh(\alpha)| \leq 1$, the modulus of the integrand in (81) is bounded by $|J_2(\zeta/\ell)|/\zeta$. As $|J_2(\zeta/\ell)|$ falls off like $\zeta^{-1/2}$ for $\zeta \to \infty$, this guarantees absolute convergence of the integral. □

Example 3: A worldline with diverging self-force integral

From Propositions 2 and 3 we know that the self-force is finite if the particle's worldline is bounded away from the light-cone or if it is contained in a timelike plane. Actually, the proof of Proposition 3 can be generalized to the case that the worldline, rather than being confined to P, approaches P sufficiently quickly for $\tau \to \tau_{\min}$. This leaves only a class of rather contrived motions for which the self-force integral could diverge: The worldline must approach the light-cone for $\tau \to \tau_{\min}$ with a sufficiently large tangential velocity component. In this section we present such an example for which the self-force integral, indeed, diverges at one point.

We find it convenient to give the worldline in terms of a *past-oriented* curve parameter γ which is *not* proper time,

$$\xi^a(\gamma) \;=\; \gamma\Big\{ -\sqrt{1+s(\gamma)}\,\delta^a_0 + \sqrt{s(\gamma)}\Big(\cos(\beta(\gamma))\delta^a_1 + \sin(\beta(\gamma))\delta^a_2\Big)\Big\} \tag{82}$$

where

$$s(\gamma) \;=\; \frac{\gamma}{\ell\tanh(\gamma/\ell)} \tag{83}$$

and

$$\beta(\gamma) \;=\; \frac{1}{2}\left(\sqrt{\pi}C_F(\sqrt{2s(\gamma)/\pi}) - \sqrt{\pi}S_F(\sqrt{2s(\gamma)/\pi}) - \frac{\sin(s(\gamma)) + \cos(s(\gamma))}{\sqrt{2s(\gamma)}}\right)\,. \tag{84}$$

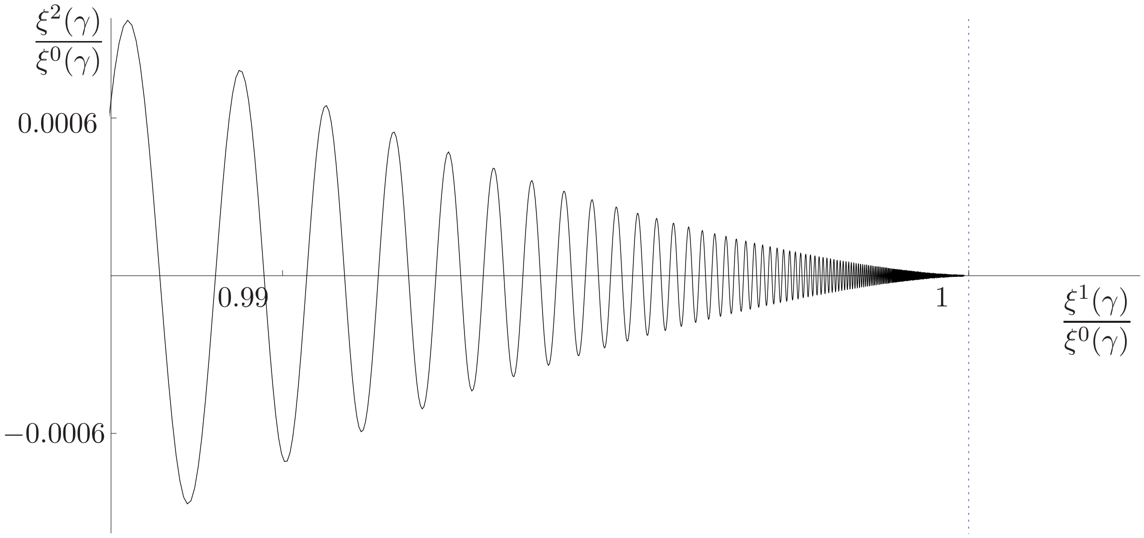

Figure 6. The worldline (82) approaches the past light-cone (dotted circle) with an oscillatory tangential velocity component

Here C_F and S_F are the Fresnel-C and Fresnel-S functions. Note that

$$\beta'(\gamma) \; = \; \frac{s'(\gamma)\sin(s(\gamma)+\pi/4)}{4\,s(\gamma)^{3/2}} \,. \tag{85}$$

$\boldsymbol{\xi}$ is an analytic timelike curve that approaches the past light-cone with an oscillatory tangential velocity component, see Fig. 6. The curve is, indeed, everywhere timelike as can be seen from

$$\begin{aligned} \eta_{ab}\frac{d\xi^a(\gamma)}{d\gamma}\frac{d\xi^b(\gamma)}{d\gamma} \; &= \; -1 + \frac{\gamma^2 s'(\gamma)^2}{4s(\gamma)(1+s(\gamma))} + \frac{\gamma^2 s'(\gamma)^2}{16 s(\gamma)^2}\sin^2(s(\gamma)+\pi/4) \\ &< \; -1 + \frac{1}{4} + \frac{1}{16} = \frac{-11}{16} \,. \end{aligned} \tag{86}$$

It can be shown that the 4-acceleration of $\boldsymbol{\xi}$ is bounded and that its future is all of Minkowski spacetime.

We show that for this worldline $\boldsymbol{\xi}$ the electromagnetic field (38) diverges on a hyperplane implying that the self-force becomes infinite at one instant. To that end we have to rewrite the integral in (38) as an integral over γ and to investigate the behavior of the integrand for $\gamma \to \infty$. We first observe that, if γ tends to ∞,

$$\begin{aligned} s(\gamma) \; &= \frac{\gamma}{\ell} + O(\gamma^{-n}) \,, \quad s'(\gamma) = \frac{\gamma}{\ell} + O(\gamma^{-n}) \,, \\ \beta'(\gamma) &= \frac{\sqrt{\ell}\sin\left(\frac{\gamma}{\ell}+\frac{\pi}{4}\right)}{\gamma^{3/2}} + O(\gamma^{-n}) \end{aligned} \tag{87}$$

for all $n \in \mathbb{N}$. Moreover, from the standard asymptotic formulas for the Fresnel functions we find

$$\cos(\beta(\gamma)) = 1 + O(\gamma^{-3}) \,, \qquad \sin(\beta(\gamma)) = O(\gamma^{-3/2}) \,. \tag{88}$$

With the help of these formulas we find that the parameter ζ which is used as the integration variable in (38) is related to our curve parameter γ by

$$\zeta^2 = -(x^a - \xi^a(\gamma))(x_a - \xi_a(\gamma)) = \gamma^2 + 2\frac{\gamma^{3/2}}{\sqrt{\ell}}(x^0 + x^1) + O(\gamma^0) \,, \tag{89}$$

hence

$$\zeta = \gamma + \sqrt{\frac{\gamma}{\ell}}(x^0 + x^1) - \frac{(x^0 + x^1)^2}{\ell} + O(\gamma^{-1/2}) \,. \tag{90}$$

Inserting this expression into the well-known asymptotic formula for the Bessel function J_2 yields

$$\begin{aligned} J_2\Big(\frac{\zeta}{\ell}\Big) &= -\sqrt{\frac{2\ell}{\pi\zeta}}\sin\Big(\frac{\zeta}{\ell} + \frac{\pi}{4}\Big) + O(\zeta^{-3/2}) \\ &= -\sqrt{\frac{2\ell}{\pi\gamma}}\sin\Big(\frac{\gamma}{\ell} + \frac{\pi}{4} + \frac{\sqrt{\gamma}(x^0 + x^1)}{\ell^{3/2}} - \frac{(x^0 + x^1)^2}{\ell^2}\Big) + O(\gamma^{-1})\,. \end{aligned} \tag{91}$$

After these preparations, we are ready to evaluate the integral in (38). If we use γ as the integration variable, writing $\gamma_R(\boldsymbol{x})$ for the parameter value that corresponds to $\tau_R(\boldsymbol{x})$ and thus to $\zeta = 0$, this integral reads

$$\begin{aligned} I_{ab} &= \int_{\gamma_R(\boldsymbol{x})}^{\infty} \frac{\Big((x_b - \xi_b(\gamma))\frac{d\xi_a(\gamma)}{d\gamma} - (x_a - \xi_a(\gamma))\frac{d\xi_b(\gamma)}{d\gamma}\Big)}{\frac{d\xi^c(\gamma)}{d\gamma}(x_c - \xi_c(\gamma))} \frac{J_2(\zeta/\ell)}{\zeta}\frac{d\zeta}{d\gamma}\,d\gamma \\ &= \int_{\gamma_R(\boldsymbol{x})}^{\infty} \frac{\Big((x_b - \xi_b(\gamma))\frac{d\xi_a(\gamma)}{d\gamma} - (x_a - \xi_a(\gamma))\frac{d\xi_b(\gamma)}{d\gamma}\Big)}{\zeta\,\frac{d\zeta}{d\gamma}} \frac{J_2(\zeta/\ell)}{\zeta}\frac{d\zeta}{d\gamma}\,d\gamma \\ &= \int_{\gamma_R(\boldsymbol{x})}^{\infty} \Big((x_b - \xi_b(\gamma))\frac{d\xi_a(\gamma)}{d\gamma} - (x_a - \xi_a(\gamma))\frac{d\xi_b(\gamma)}{d\gamma}\Big) \frac{J_2(\zeta/\ell)}{\zeta^2}\,d\gamma\,. \end{aligned} \tag{92}$$

We evaluate this equation for $a = 2$ and $b = 0$. As

$$(x_0 - \xi_0(\gamma))\frac{d\xi_2(\gamma)}{d\gamma} - (x_2 - \xi_2(\gamma))\frac{d\xi_0(\gamma)}{d\gamma} = \frac{-\gamma^{3/2}}{4\sqrt{\ell}}\sin\Big(\frac{\gamma}{\ell} + \frac{\pi}{4}\Big) + O(\gamma^{1/2})\,, \tag{93}$$

we find with our asymptotic formula for $J_2(\zeta/\ell)$ from above

$$\begin{aligned} I_{20} &= \int_{\gamma_R(\boldsymbol{x})}^{\infty} \left\{ \frac{\ell\sin\Big(\frac{\gamma}{\ell} + \frac{\pi}{4}\Big)}{2\sqrt{2\pi}\,\gamma}\sin\Big(\frac{\gamma}{\ell} + \frac{\pi}{4} + \frac{\sqrt{\gamma}(x^0 + x^1)}{\ell^{3/2}} - \frac{(x^0 + x^1)^2}{\ell^2}\Big) + O(\gamma^{-3/2}) \right\} d\gamma \\ &= \frac{\ell}{4\sqrt{2\pi}}\int_{\gamma_R(\boldsymbol{x})}^{\infty} \Big\{ \cos\Big(\frac{\sqrt{\gamma}(x^0 + x^1)}{\ell^{3/2}} - \frac{(x^0 + x^1)^2}{\ell^2}\Big) \\ &\qquad\qquad - \cos\Big(\frac{2\gamma}{\ell} + \frac{\pi}{2} + \frac{\sqrt{\gamma}(x^0 + x^1)}{\ell^{3/2}} - \frac{(x^0 + x^1)^2}{\ell^2}\Big)\Big\}\frac{d\gamma}{\gamma} + \ldots \end{aligned} \tag{94}$$

Here and in the following, the ellipses indicate a term that is finite for all $\boldsymbol{x}$. The integral over the second term is finite for all $\boldsymbol{x}$. If we decompose the remaining integral into an integration from $\gamma_R(\boldsymbol{x})$ to ℓ and an integration from ℓ to infinity we find

$$\begin{aligned} I_{20} &= \frac{\ell}{4\sqrt{2\pi}}\int_{\ell}^{\infty} \cos\Big(\frac{\sqrt{\gamma}(x^0 + x^1)}{\ell^{3/2}} - \frac{(x^0 + x^1)^2}{\ell^2}\Big)\frac{d\gamma}{\gamma} + \ldots \\ &= \frac{\ell}{4\sqrt{2\pi}}\left\{ -2\cos\left(\frac{(x^0 + x^1)^2}{\ell^2}\right)\,\mathrm{Ci}\left(\frac{|x^0 + x^1|^{3/2}}{\ell^{3/2}}\right) \right. \\ &\qquad\qquad \left. - \sin\left(\frac{(x^0 + x^1)^2}{\ell^2}\right)\left(\pi - 2\,\mathrm{Si}\left(\frac{|x^0 + x^1|^{3/2}}{\ell^{3/2}}\right)\right)\right\} + \ldots \end{aligned} \tag{95}$$

Here Ci and Si denote the cosine integral and the sine integral, respectively. As the cosine integral diverges logarithmically if its argument approaches zero, we have found that the (20)-component of the electromagnetic field according to (38) is given by

$$F_{20}(\boldsymbol{x}) \;=\; -\frac{3q}{4\sqrt{2\pi}\ell}\log\left(\frac{|x^0+x^1|}{\ell}\right)\;+\;\dots\,, \tag{96}$$

i.e., that this field component diverges on the lightlike hyperplane $x^0+x^1=0$. This may be interpreted as a shock front propagating at the speed of light. The same divergence is found, by a completely analogous calculation, for the component $F_{21}=-F_{12}$,

$$F_{21}(\boldsymbol{x}) \;=\; -\frac{3q}{4\sqrt{2\pi}\ell}\log\left(\frac{|x^0+x^1|}{\ell}\right)\;+\;\dots\,, \tag{97}$$

while all other components are finite everywhere. As a consequence, the self-force becomes infinite at the instant when the charged particle crosses the hypersurface $x^0+x^1=0$ which happens at the origin of the coordinate system. This means that at this instant an infinite external Minkowski force is necessary to keep the particle on its prescribed worldline.

Note that the divergence is logarithmic in the neighborhood of a lightlike hyperplane and thus rather mild, since any timelike worldline crosses it at most once. It is true that a charged particle would experience an infinite relativistic Lorentz force at the instant when it crosses the hypersurface $x^0+x^1=0$. However, as $\int^y \log|x|\,dx$ is finite-valued and continuous at $y=0$, the particle's velocity would still be finite-valued and continuous, i.e., the particle's worldline would still be a C^1 curve.

6. The Abraham-Lorentz-Dirac limit

In the standard Maxwell-Lorentz theory with point charges, i.e., for $\ell\to 0$, the self-force becomes infinite in (54). Dirac's solution to give a meaning to the equation of motion (55) in this case was to assume that the inertial mass became negative infinite in order to cancel the infinite contribution from the self-force. After this cancellation, one ends up with the Abraham-Lorentz-Dirac equation which involves a renormalized (or dressed) mass which is positive and finite. It is interesting to see how the Abraham-Lorentz-Dirac equation is reproduced from the Bopp-Podolsky theory in the limit $\ell\to 0$. To that end we substitute in (56) the integration variable $\zeta=\ell\sigma$,

$$f_a^{\mathrm{S}}(\tau_0) \;=\; \frac{q^2\dot{\xi}^b(\tau_0)}{\ell}\int_0^\infty \frac{\partial W_{ab}}{\partial\zeta}(\ell\sigma,\boldsymbol{\xi}(\tau_0))\frac{d^2\chi(\sigma)}{d\sigma^2}\,d\sigma \tag{98}$$

where

$$\chi(\sigma)=\int_\sigma^\infty\left(\int_{\sigma'}^\infty\frac{J_1(\sigma'')}{\sigma''}d\sigma''\right)d\sigma'\,. \tag{99}$$

With

$$\chi(0)=1\,,\qquad \chi'(0)=-1 \tag{100}$$

two times integrating by parts yields

$$f^{\mathrm{S}}_a(\tau_0) \,=\, -\,\frac{q^2}{2\ell}\ddot{\xi}_a(\tau_0) \,+\, \frac{2}{3}\Big(\,\dddot{\xi}_a(\tau_0) \,+\dot{\xi}_a(\tau_0)\,\dot{\xi}^b(\tau_0)\,\dddot{\xi}_b\,(\tau_0)\Big) \qquad (101)$$
$$+\,\ell q^2\dot{\xi}^b(\tau_0)\int_0^\infty \frac{\partial^3 W_{ab}}{\partial\zeta^3}(\ell\sigma, \boldsymbol{\xi}(\tau_0))\,\chi(\sigma)\,d\sigma\,.$$

The first term diverges for $\ell \to 0$. Following Dirac's idea of mass renormalization, the parameter m must depend on ℓ and become negative infinite such that the "dressed mass"

$$\hat{m} \,=\, \lim_{\ell\to 0}\Big(m(\ell) + \frac{q^2}{2\ell}\Big) \qquad (102)$$

remains finite and positive. In this limit, the equation of motion reads

$$\hat{m}\,\ddot{\xi}_a(\tau) = \frac{2q^2}{3}\Big(\,\dddot{\xi}_a\,(\tau) + \dot{\xi}_a(\tau)\dot{\xi}^b(\tau)\,\dddot{\xi}_b\,(\tau)\Big) \,+\, f^{\mathrm{e}}_a(\tau) \qquad (103)$$
$$+\,\lim_{\ell\to 0}\Bigg(\ell q^2\dot{\xi}^b(\tau_0)\int_0^\infty \frac{\partial^3 W_{ab}}{\partial\zeta^3}(\ell\sigma, \boldsymbol{\xi}(\tau_0))\,\chi(\sigma)\,d\sigma\Bigg)\,.$$

If the integral is bounded, the last term vanishes and we get the Abraham-Lorentz-Dirac equation. From (99) we find, with the help of the well-known asymptotic formula for the Bessel function J_1, that $\chi(\sigma) = O(\sigma^{-3/2})$ for $\sigma \to \infty$. So the integral in (103) is certainly bounded if $\partial^3 W_{ab}/\partial\zeta^3$ is bounded for $\zeta \to \infty$. A sufficient (but not necessary) condition is that ξ is bounded away from the light-cone and that all components $\dot{\xi}^a$, $\ddot{\xi}^a$, $\dddot{\xi}^a$, and $\ddddot{\xi}^a$ are bounded.

7. Conclusions

In this paper we have demonstrated that in the Bopp-Podolsky theory a self-force of a charged point particle can be defined by an integral that is absolutely convergent for a large class of worldlines on Minkowski spacetime. We have also provided a (contrived) example where its electromagnetic field diverges on a lightlike plane, so the self-force diverges at one point of the worldline. However, even in this case the electromagnetic field is a locally integrable function (i.e., a regular distribution), yielding a C^1 solution to the equation of motion (55). This is to be contrasted with the standard Maxwell-Lorentz theory in vacuo where the self-field of a charged point particle is infinite at every point of the particle's worldline, and the singularity is so bad that the field energy in an arbitrarily small ball around the charge is infinite necessitating classical mass renormalization. In the Bopp-Podolsky theory there is no need for mass renormalization; the equation of motion (55) is an integro-differential equation making sense with a finite inertial mass m.

It should also be emphasized that in the Born-Infeld theory, which is to be viewed as a natural rival to the Bopp-Podolsky theory, virtually nothing is known about finiteness of the self-force of an accelerated particle whose worldline may approach the light-cone. So it seems fair to say that, at least in view of the motion of charged point particles, the Bopp-Podolsky theory is in a more promising state.

Some questions remain open. It is clearly important to establish a strategy for solving (55). This may include formulating a consistent Cauchy problem that may include either background or dynamic fields satisfying the Bopp-Podolsky field equations coupled to the particle. Furthermore, it would be desirable to have a proof that *all* worldlines solving (55) are at least C^1. It is also important to demonstrate that the equation of motion (55) with vanishing external Minkowski force is free of run-away solutions. Partial results in this direction have been found by Frenkel & Santos (1999), but a general proof is still missing. If these questions can be satisfactorily resolved, the Bopp-Podolsky theory offers a physically acceptable framework in which to explore the classical electromagnetic back-reaction on charged point particles.

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Appendix

The formulation of a stress-energy-momentum tensor for the theory discussed in this paper appears to have had a chequered history. Bopp (1940) writes down a stress-energy-momentum tensor that is obviously based on the decomposition (10) of the potential, but no derivation is given. Podolsky (1942) also writes down a stress-energy-momentum tensor with a promise to derive it in (Podolsky & Kikuchi 1944) from arguments based on a canonical approach. In our view this did not succeed and the further derivation in (Podolsky & Schwed 1948) lacked transparency. To our knowledge there has been no subsequent attempt to derive any stress-energy-momentum tensor appropriate to the theory under discussion. In view of these comments it may be of value to put the matter into a modern perspective by offering a derivation based on metric variations of the Bopp-Podolsky action. This requires formulating the theory on a *curved* spacetime manifold.

The natural tools for this purpose exploit the exterior calculus of differential forms using properties of the Hodge map $\star$ associated with the spacetime metric g and the nilpotency of the exterior derivative d. For background material on exterior calculus we refer to Straumann (1984) whose sign and factor conventions we adopt. The Bopp-

Podolsky action reads

$$S[A,J,g]=\int_{\mathcal{M}}\Lambda=\int_{\mathcal{M}}(\Lambda_{\rm EM}-A\wedge J) \tag{104}$$

with

$$\Lambda_{\rm EM}=\frac{1}{8\pi}F\wedge\star F-\frac{\ell^2}{8\pi}G\wedge\star G \tag{105}$$

where $F=dA$ and $G=d\star F$ are smooth. Here we assume that J is a prescribed (non-dynamical) smooth current 3-form that satisfies the conservation law $dJ=0$ on $\mathcal{M}$. In the body of the paper we have restricted ourselves to the case of a flat metric and used inertial coordinates. Then the action (104) reduces to (4) where the current 4-vector $j=j^a\partial_a$ is related to the current 3-form by $g(j,\,\cdot\,)=\star J$.

A direct route to the symmetric dynamical (Hilbert) stress-energy-momentum tensor, associated with $\Lambda_{\rm EM}$, is obtained by making compact variations of the metric tensor in $\Lambda_{\rm EM}$. Such variations can be induced by making independent variations $\dot{e}^a$ in a local $g-$orthonormal coframe $\{e^a\}$, $a=0,1,2,3$ since in such a basis $g=\eta_{ab}e^a\otimes e^b$. Such variations give rise to a set of 3-forms τ_a defined by

$$\int_{\mathcal{M}}\dot{\Lambda}_{\rm EM}=\int_{\mathcal{M}}\dot{e}^a\wedge\tau_a \tag{106}$$

and a stress-energy-momentum tensor $T=T_{ab}e^a\otimes e^b$ with components

$$T_{ab}=\eta_{bc}\star(\tau_a\wedge e^c)\,. \tag{107}$$

The covariant divergence of T then follows as

$$\nabla\cdot T=(\star^{-1}D\tau_a)e^a \tag{108}$$

where D denotes the covariant exterior derivative, see e.g. Benn & Tucker (1988). Since T is symmetric

$$D\tau_a=d\tau_a-i_a de^b\wedge\tau_b\ . \tag{109}$$

If we make compact variations of the potential A in S, rather than of the metric, we obtain the field equation of the Bopp-Podolsky theory. We can derive the τ_a and the field equation in one go if we allow for partial variations of the potential and of the metric simultaneously. (Note that the current J is assumed to be given and is kept fixed during the variation.) Then the total variation of the Lagrangian 4-form is written

$$\Lambda^{\bullet}=\frac{1}{8\pi}(F\wedge\star F)^{\bullet}-\frac{\ell^2}{8\pi}(G\wedge\star G)^{\bullet}-A^{\bullet}\wedge J\,. \tag{110}$$

For calculating the extremum of Λ we use the formula (Dereli et al. 2007)

$$(\star\Psi)^{\bullet}=e_c^{\bullet}\wedge i^c(\star\Psi)-\star(e_c^{\bullet}\wedge i^c\Psi)+\star\Psi^{\bullet} \tag{111}$$

that holds for any $p-$form Ψ, where i_c denotes the contraction operator (or interior derivative) with respect to the vector field X_c defined by $e^a(X_c)=\delta^a_c$. Moreover, we use standard rules of exterior calculus, such as $\Psi\wedge\star\Phi=\Phi\wedge\star\Psi$ for any $p-$forms Ψ,Φ, the

graded derivative property and nilpotency of d, and the commutativity of d with the variations. Thus

$$(F\wedge\star F)^{\bullet}=2\,d(A^{\bullet}\wedge\star F)+2\,A^{\bullet}\wedge d\star F+e_c^{\bullet}\wedge(F\wedge i^c\star F-i^cF\wedge F)\tag{112}$$

and

$$\begin{aligned}(G\wedge\star G)^{\bullet}=2\,d\Big(\Big(e_c^{\bullet}\wedge i^c\star F-\star(e_c^{\bullet}\wedge i^cF)+\star F^{\bullet}\Big)\wedge\star G\Big)\\ -2\,A^{\bullet}\wedge d\star d\star G-e_c^{\bullet}\wedge\Big(2\,i^c\star F\wedge d\star G-2\,i^cF\wedge\star d\star G\\ +G\wedge i^c\star G+i^cG\wedge\star G\Big)\,.\end{aligned}\tag{113}$$

Inserting (112) and (113) into (110) yields

$$\Lambda^{\bullet}=d\Phi+e_c^{\bullet}\wedge\tau_c+A^{\bullet}\wedge\frac{1}{4\pi}\,dH-A^{\bullet}\wedge J\tag{114}$$

where

$$4\pi\Phi=A^{\bullet}\wedge H-\ell^2\Big(e_c^{\bullet}\wedge i^c\star F-\star(e_c^{\bullet}\wedge i^cF)+\star F^{\bullet}\Big)\wedge\star G\,,\tag{115}$$

$$H=\star F+\ell^2\star d\star d\star F\tag{116}$$

and

$$8\pi\tau_c=F\wedge i_c\star F-i_cF\wedge\star F+\ell^2\Big(G\wedge i_c\star G+i_cG\wedge\star G-2i_cF\wedge\star d\star G+2i_c\star F\wedge d\star G\Big)\,.\tag{117}$$

The field equations are determined by requiring that the action is stationary for partial variations of the potential only (i.e., $\dot{e}^c=0$) that are compactly supported (i.e $\int_{\mathcal{M}}d\Phi=0$):

$$dH=4\pi J\,.\tag{118}$$

Similarly for partial variations with $\dot{A}=0$, the τ_c from (117) give, via (107), the dynamical stress-energy-momentum tensor of the Bopp-Podolsky theory which is automatically symmetric. A lengthy but routine calculation of $D\tau_a$ then shows, with the aid of the field equations (118), that the divergence (108) yields the relativistic Lorentz force,

$$\nabla\cdot T=F(\,\cdot\,,j)\,.\tag{119}$$

A less lengthy approach to derive (119) can be based on the use of a one-parameter family of diffeomorphisms on the spacetime domain $\mathcal{M}$ generated by any compactly supported vector field X. Then in terms of the Lie derivative L_X

$$L_X\Lambda_{\rm EM}=d\Phi_X+L_Xe^c\wedge\tau_c+\frac{1}{4\pi}L_XA\wedge dH\tag{120}$$

with the same τ_c and the same H as in (114) and

$$4\pi\Phi_X=L_XA\wedge H-\ell^2\Big(L_Xe^c\wedge i_c\star F-\star(L_Xe^c\wedge i_cF)+\star L_XF\Big)\wedge\star G\,.\tag{121}$$

For the derivation of (120) we have used the identity

$$L_X(\star\Psi)=L_Xe^c\wedge i_c(\star\Psi)-\star(L_Xe^c\wedge i_c\Psi)+\star L_X\Psi\tag{122}$$

which holds for any p-form Ψ on M. With the relations $L_X = i_X d + d i_X$ and $d\Lambda_{\rm EM} = 0$ (since $\Lambda_{\rm EM}$ is a 4-form on spacetime), (120) yields the identity

$$d\,\alpha_X = \beta_X \tag{123}$$

where

$$\alpha_X = i_X\Lambda_{\rm EM} - i_X e^c \wedge \tau_c - i_X A \wedge \frac{1}{4\pi} dH - \Phi_X \tag{124}$$

and

$$\beta_X = i_X de^c \wedge \tau_c - i_X e^c \wedge d\tau_c + i_X F \wedge \frac{1}{4\pi} dH\,. \tag{125}$$

Integrating (123) over an open domain U containing the support of X and using Stokes's theorem yields

$$\int_U \beta_X = \int_U d\alpha_X = \int_{\partial U} \alpha_X = 0$$

since $\alpha_X|_{\partial U} = 0$. Furthermore since β_X has the linearity property $\beta_{fX} = f\,\beta_X$ for any smooth function f, it follows♯ that $\beta_X = 0$ (Gratus et al. 2012). Finally choosing $X = X_a$ and using the field equation (118) together with (109) gives

$$D\tau_a = F(X_a, \cdot\,) \wedge J \tag{126}$$

which is equivalent to (119), using (108). From this derivation one concludes that, for *any* diffeomorphism and gauge invariant action

$$S[A, J, g] = \int_{\mathcal{M}} \Lambda(A, J, g)$$

constructed from

$$\Lambda(A, J, g) = \Lambda^{\rm f}(F, g) - A \wedge J \tag{127}$$

and $F = dA$ equation (126) is satisfied when J and g are background fields with $\Lambda^{\rm f}(F, g)$ arbitrary. However for consistency this requires that J be a prescribed exact 3-form. On a topologically trivial spacetime domain $\mathcal{M}$ this is implied by $dJ = 0$.

If the background metric is flat and inertial coordinates are used, (118) reduces to (5); in this case the components of the stress-energy-momentum tensor associated with (117) are

$$\begin{aligned}4\pi T_{cd} = \frac{1}{4} F_{ab}F^{ab}\eta_{cd} - F_{db}F_c{}^b + \ell^2\Big(\partial^b F_{bc}\partial^a F_{ad} - \frac{1}{2}\partial^a F_{ab}\partial_\ell F^{\ell b}\eta_{cd}\Big) \\ - \ell^2\Big(F_{cb}\partial^a\partial_a F^b{}_d + F_{db}\partial^b\partial^a F_{ac} + F_{cb}\partial^b\partial^a F_{ad} + F^{ab}\partial^e\partial_e F_{ab}\eta_{cd}\Big)\end{aligned} \tag{128}$$

where we have used $dF = 0$. In this flat metric with the field equations (5), the stress-energy-momentum tensor (128) satisfies $\partial^c T_{cd} = F_{db}j^b$ which is the flat-space coordinate version of (119). The stress-energy-momentum tensor (128) coincides with that written down by Podolsky (1942) and subsequently quoted by Zayats (2014). The *derivation* above shows that, in general, it coincides with the Hilbert stress-energy-momentum tensor derived from metric variations of the action (104).

♯ Note however since $\alpha_{fX} \neq f\,\alpha_X$ for arbitrary f, one cannot similarly conclude that $\alpha_X = 0$.

Finally we point out how different definitions of the stress-energy-momentum tensor for the Bopp-Podolsky theory are responsible for the historic problems outlined in the beginning of this appendix. In general (9) becomes

$$\tilde{A} = -\ell^2 \star G\,, \quad \hat{A} = A - \ell^2 \star G\,. \tag{129}$$

and the Lagrangian (104) can be rewritten as

$$\Lambda = \Lambda^1 + \frac{\ell^2}{4\pi}\, d(\star\, d \star F \wedge \star F) \tag{130}$$

where

$$\Lambda^1 = \frac{1}{8\pi} d\hat{A} \wedge \star d\hat{A} - \frac{1}{8\pi}\left(d\tilde{A} \wedge \star d\tilde{A} + \frac{1}{\ell^2}\tilde{A} \wedge \star\tilde{A}\right) - (\hat{A} - \tilde{A}) \wedge J\,. \tag{131}$$

As Λ and Λ^1 differ only by an exact form, they lead to the same field equations and to the same *Hilbert* stress-energy-momentum tensor. Indeed, varying the action

$$\begin{aligned} S^1[\hat{A}, \tilde{A}, g] &= \int_{\mathcal{M}} \Lambda^1 \\ &= \int_{\mathcal{M}} \Big(\frac{1}{8\pi} d\hat{A} \wedge \star d\hat{A} - \frac{1}{8\pi}\left(d\tilde{A} \wedge \star d\tilde{A} + \frac{1}{\ell^2}\tilde{A} \wedge \star\tilde{A}\right) - (\hat{A} - \tilde{A}) \wedge J\Big) \end{aligned} \tag{132}$$

with respect to $\hat{A}$ and $\tilde{A}$ respectively yields the field equations

$$d \star d\hat{A} = 4\pi J\,, \quad d \star d\tilde{A} - \frac{1}{\ell^2} \star \tilde{A} = 4\pi J\,. \tag{133}$$

These equations imply that $\tilde{A}$ necessarily satisfies the Lorenz gauge condition, $d\star\tilde{A} = 0$.

The field equations (133) reduce to (7) and (8) in flat spacetime using inertial coordinates and the Lorenz gauge condition on $\hat{A}$.

The $g-$orthonormal co-frame variation of Λ^1 gives rise to the Hilbert stress forms

$$8\pi\tau_c = d\hat{A}\wedge i_c\star d\hat{A} - i_c d\hat{A}\wedge\star d\hat{A} - d\tilde{A}\wedge i_c\star d\tilde{A} + i_c d\tilde{A}\wedge\star d\tilde{A} + \frac{1}{\ell^2}\left(\tilde{A}\wedge i_c\star\tilde{A} + i_c\tilde{A}\wedge\star\tilde{A}\right). \tag{134}$$

Substituting from (129), we see that these stress forms indeed coincide with (117). On flat spacetime in inertial coordinates, (134) yields the stress-energy-momentum tensor

$$4\pi T_{cd} = \frac{1}{4}\hat{F}_{ab}\hat{F}^{ab}\eta_{cd} - \hat{F}_{ac}\hat{F}^{a}{}_{d} - \frac{1}{4}\tilde{F}_{ab}\tilde{F}^{ab}\eta_{cd} + \tilde{F}_{ac}\tilde{F}^{a}{}_{d} + \frac{1}{\ell^2}\left(\tilde{A}_c\tilde{A}_d - \frac{1}{2}\tilde{A}_a\tilde{A}^a\eta_{cd}\right) \tag{135}$$

where $\hat{F}_{ab} = \partial_a\hat{A}_b - \partial_b\hat{A}_a$ and $\tilde{F}_{ab} = \partial_a\tilde{A}_b - \partial_b\tilde{A}_a$. This is the stress-energy-momentum tensor given by Bopp (1940), cf. again Zayats (2014).

Furthermore the same stress forms (134) arise using the canonical Belinfante-Rosenfeld symmetrization of the Noether current associated with the Lagrangian (131). By contrast the canonical Belinfante-Rosenfeld procedure applied to the action (2) does not yield the stress-energy-momentum forms (134). This illustrates the fact that two Lagrangians that differ by an exact form, while yielding the same Hilbert stress-energy-momentum tensors, in general, yield different stress-energy-momentum tensors following the canonical Belinfante-Rosenfeld procedure.

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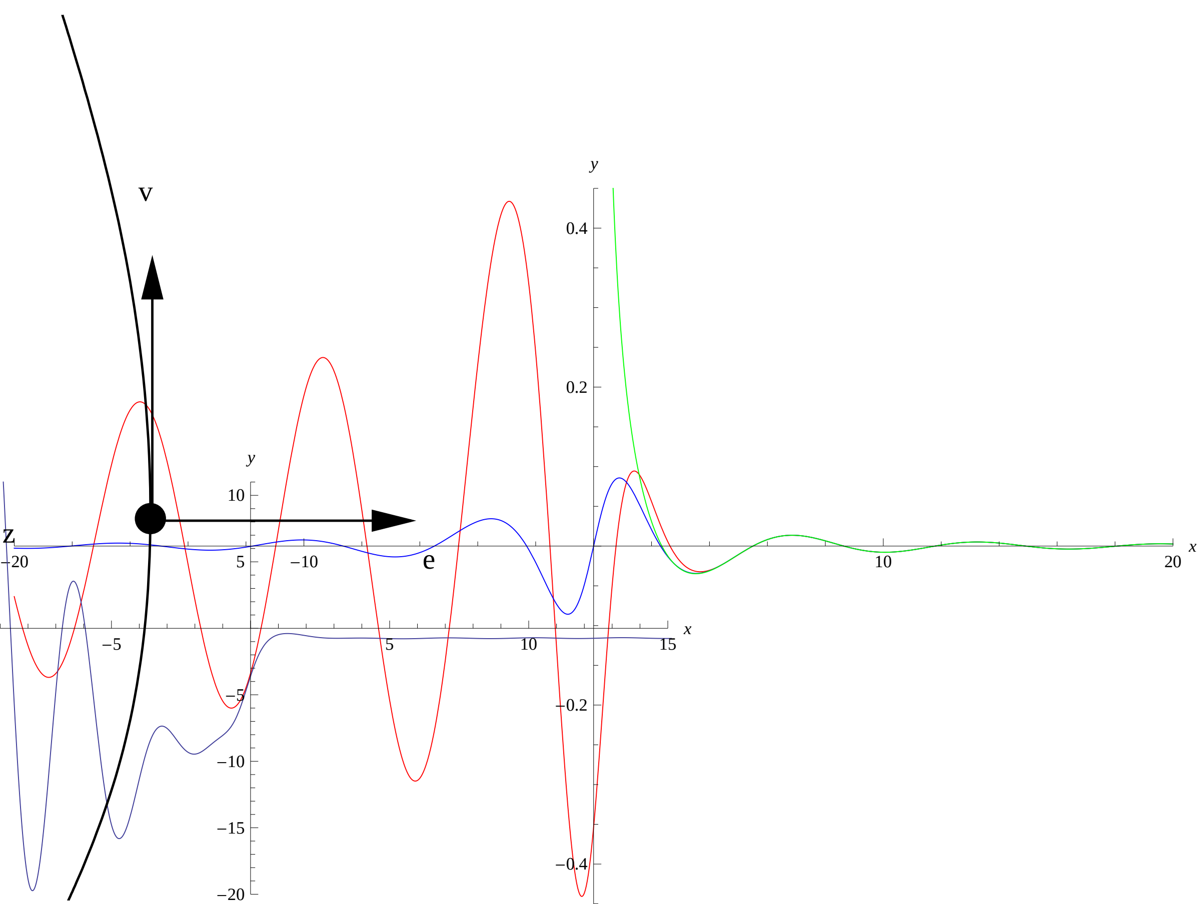

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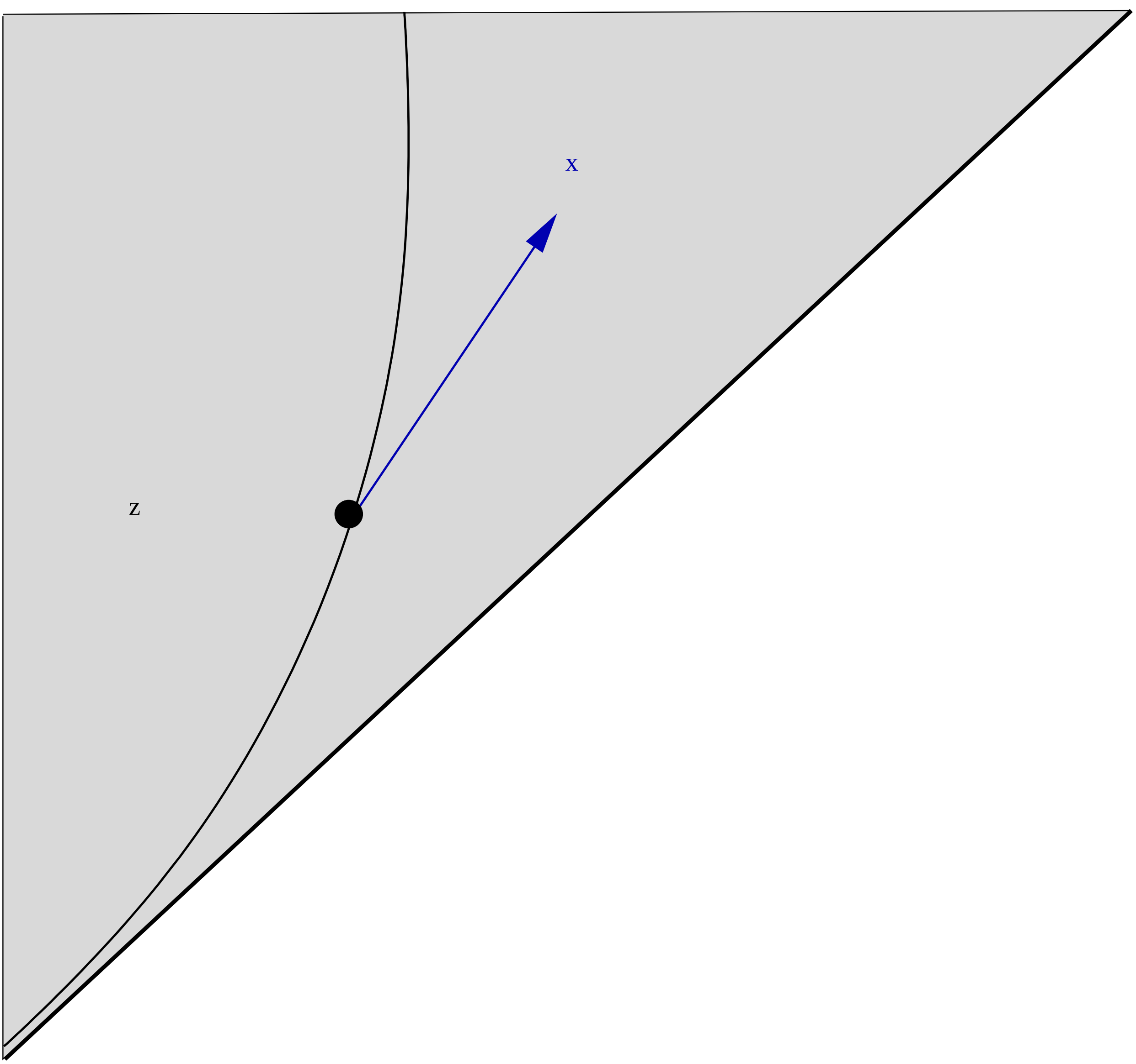
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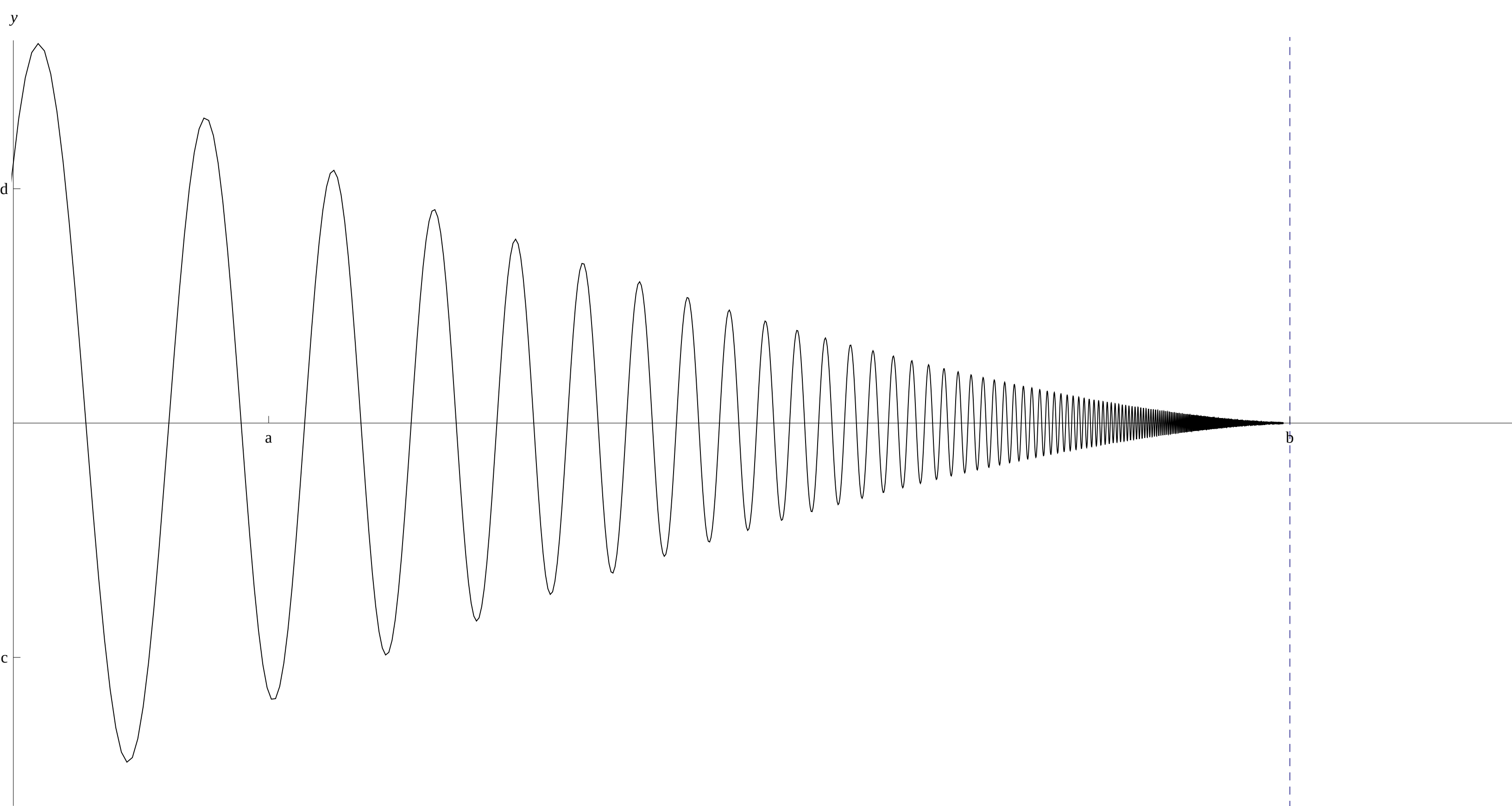

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